Review

- $h(t) = \int_{-\infty}^{+\infty} H(f) e^{j2p\,ft} df \xrightarrow{FT} H(f) = \int_{-\infty}^{+\infty} h(t) e^{-j2p\,ft} dt$

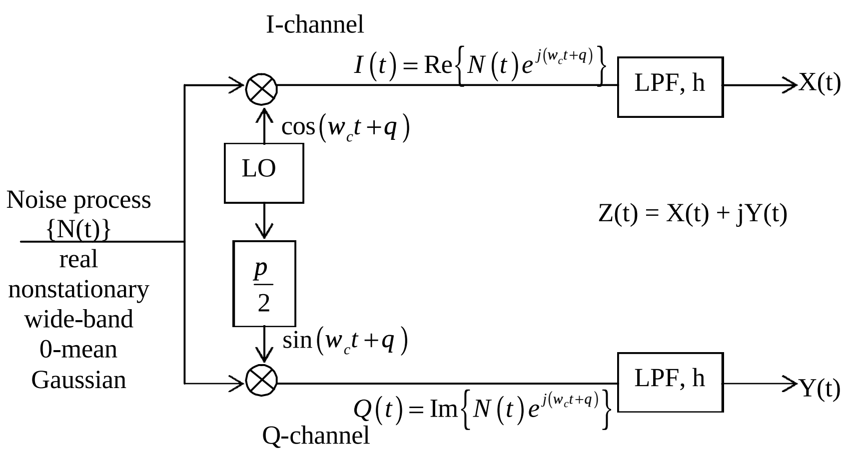
- Let $y(t) = w(t) \otimes h(t) = \int_{-\infty}^{+\infty} w(t)h(t-t)dt$

  For $\tilde{w}(t) = w(t-a)$,

  $$\tilde{w} \otimes h(t) = \int_{-\infty}^{+\infty} w\underbrace{(t-a)}_{m} h(t-t)dt = \int_{-\infty}^{+\infty} w(m)h\big(t-(m+a)\big)dm$$

  $$= \int_{-\infty}^{+\infty} w(m)h\big((t-a)-m\big)dm = y(t-a) = \tilde{y}(t)$$

Heterodyne Outputs are circularly Gaussian



- Given
  - N(t) : Noise Process, real ,non-stationary, broadband, 0-mean, Gaussian
  - LPF is lowpass in the sense that,
    - for any fixed *u*,
      - *h(t,u)* varies with t orders of magnitude more slowly than does a sinewave at the carrier frequency f_c.

  Then, {Z(t) = X(t) + jY(t)} is
  - 0-mean proper complex Gaussian random process (E*Z(s)Z(t) = 0*)
  - distributed in a manner that is independent of the LO phase $\hat{f}$

  - $I(t) = N(t)\cos(w_c t + q) = \mathrm{Re}\{N(t)e^{j(w_c t+q)}\}$
    - {I(t)} is Gaussian, but not w.s.s.

- Gaussian because I(t) = N(t) times a deterministic constant depends only on time t. Thus, $\underline{I}_t = A\underline{N}_t$. Since $\underline{N}_t$ is Gaussian, $\underline{I}_t$ is Gaussian.
- Not w.s.s., because $R_{I,I}(s,t)$ does not depends only on t-s

- $Q(t) = N(t)\sin(w_c t + q) = \mathrm{Im}\left\{N(t)e^{j(w_c t+q)}\right\}$
  - Similarly, Gaussian, but not w.s.s.
- $C(t) = N(t)e^{j(w_c t+q)} = I(t) + jQ(t)$ is 0-mean, complex, and Gaussian

  because {N(t)} is zero-mean Gaussian.
  - $EC(t) = (EN(t))e^{j(w_c t+q)} = 0 \cdot e^{j(w_c t+q)} = 0$
  - {C(t)} is w.s.s. if {N(t)} is w.s.s.

$$R_C(s,t) = EC(s)\overline{C(t)} = EN(s)e^{j(w_c s+q)}N(t)e^{-j(w_c t+q)}$$
$$= (EN(s)N(t))e^{jw_c(s-t)}$$
$$= R_N(s,t)e^{-jw_c(t-s)}$$

If N(t) is w.s.s.,

$$R_C(s,t) = R_N(t-s)e^{-jw_c(t-s)} = R_C(t-s)$$

- $X(t) = \int_{-\infty}^{\infty} I(s)h(s,t)ds = \int_{-\infty}^{\infty} N(s)\cos(w_c s + q)h(s,t)ds$

  $Y(t) = \int_{-\infty}^{\infty} Q(s)h(s,t)ds = \int_{-\infty}^{\infty} N(s)\sin(w_c s + q)h(s,t)ds$

- Since {Z(t)} is the result of linearly filtering {C(t)},
  {Z(t)} is 0-mean, complex, Gaussian process.
- $EZ(u)Z(v) = 0$

$$Z(u) = \int I(s)h(s,u)ds + j\int Q(s)h(s,u)ds$$
$$= \int (I(s) + jQ(s))h(s,u)ds = \int C(s)h(s,u)ds$$
$$= \int N(s)e^{j(w_c s+q)}h(s,u)ds$$

Similarly, $Z(v) = \int N(t)e^{j(w_c t+q)}h(t,v)dt$

$$EZ(u)Z(v) = E\int N(s)e^{j(w_c s+q)}h(s,u)ds\int N(t)h(t,v)dt$$
$$= \iint h(s,u)h(t,v)e^{j(w_c(s+t)+2q)}EN(s)N(t)dsdt$$
$$= \iint h(s,u)h(t,v)e^{j(w_c(s+t)+2q)}R_{NN}(s,t)dsdt$$

- Assume white noise :

$$R_{NN}(s,t) = N_0 d(t-s)$$

$S_N(f) = N_0$ for all -¥ < f < ¥

$$EZ(u)Z(v) = \iint h(s,u)h(t,v)e^{j(w_c(s+t)+2q)}N_0 d(t-s)\,dsdt$$

$$= \int h(t,u)h(t,v)e^{j(w_c(t+t)+2q)}N_0 dt$$

- Assume LPF: h is lowpass in the sense of passing only frequency content much smaller than w_c

  For any fixed u and v, h(t,u)h(t,v) varies slowly with t in comparison with the real and the imaginary parts of $e^{j2w_c t}$, both of which oscillate rapidly and symmetrically about zero.

  Thus, $EZ(u)Z(v) \approx 0$ for all u and v

- Distribution of {z(t)} does not depend on the value of the LO phase offset θ.

  Let {$Z_0(t)$} be {Z(t)} in the case θ = 0.

  {Z(t)} is proper Gaussian for any value of θ

  For any given θ, consider the process {W(t)} defined by

  $$W(t) = e^{jq}Z_0(t)$$

  Clearly, {W(t)} is a complex Gaussian process.

  Moreover,

  $$EW(t) = e^{jq}EZ_0(t) = 0 = EZ_0(t)$$

  $$K_W(s,t) = EW(s)\overline{W(t)} = Ee^{jq}Z_0(s)e^{-jq}\overline{Z_0(t)} = EZ_0(s)\overline{Z_0(t)} = K_Z(s,t)$$

  So, {W(t)} is distributed exactly the way {Z_0(t)} is.

  Also,

  $$EW(s)W(t) = Ee^{jq}Z_0(s)e^{jq}Z_0(t) = e^{j2q}EZ_0(s)Z_0(t) = 0$$

  So, {W(t)} is proper.

  $$Z(t) = \int N(s)e^{j(w_c s+q)}h(s,t)ds = e^{jq}\int N(s)e^{jw_c s}h(s,t)ds = e^{jq}Z_0(t)$$

  Thus, {Z(t)} for general θ is just a phase-shifted version of the circularly Gaussian process {Z_0(t)}, so the joint distribution {Z(t)} does not depend on the choice of θ.

- Now suppose the noise input {N(t)} is not necessarily white and stationary,

  but continue to assume that typical realization of {N(t)} vary rapidly with respect to a sinewave at the carrier frequency f_c.

  Then, the noise autocorrelation function R_N(s,t) is small unless |t-s|f_c << 1.

  $$EZ(u)Z(v) = \iint h(s,u)h(t,v)e^{j(w_c(s+t)+2q)}R_{NN}(s,t)\,dsdt$$

  We can replace the factor $h(s,u)h(t,v)e^{j(w_c(s+t)+2q)}$ by $h(t,u)h(t,v)e^{j(2w_c t+2q)}$ without materially affecting the result.

  It follows that

$$EZ(u)Z(v) = \int_{-\infty}^{\infty} h(t,u)h(t,v)e^{j(2w_c t+2q)}R_N(t)\,dt$$

where $R_N(t) = \int_{-\infty}^{\infty} R_N(s,t)\,ds$.

We therefore need assume only that the function $R_N(t)$ defined by the preceding equation varies slowly with respect to $e^{jw_c t}$ in order to preserve the desired conclusion that $EZ(u)Z(v) \approx 0$ for all u and v.

This will be the case provided any and all underlying sources of nonstationariness in the input noise vary slowly in comparison with a sinewave at the carrier frequency.

QAM Communications – Quadrature Amplitude Modulation

- QAM is a digital transmission technique which conveys data at a rate of *m*-bits per symbol by sending one of $M = 2^m$ symbols during each of a succession of band intervals of duration T.
- The *M* points are arranged in a **constellation** in the (x,y)-plane
- The k^{th} point in the constellation may be described either by its Cartesian coordinates (x_k, y_k) or by its polar coordinates (r_k, q_k)
- Conflicting goals: want to arrange the points in the constellation so that
  - They are close to the origin on average so that it does not require much energy to send them
  - No two of them are close enough together that the channel noise often causes us to mistake them for one another.
- Goal: find the conditional probability of transmission for each symbol in the constellation during a baud given the data received therein, and then compute therefrom maximum likelihood ratio combining metrics on a bit-by-bit inorder to enable optimum soft-decision decoding.
- Let
  - transmitter carrier $\Rightarrow \cos(w_c t + f)$
  - receiver LO $\Rightarrow \cos\left(w_c t + \hat{f}\right)$
- transmitted signal

  = I-channel signal + Q-channel signal

  = $A(t)\cos(w_c t + f) + B(t)\sin(w_c t + f)$

  where $A(t) = \sum_k A_k g_T(t - kT)$

  $B(t) = \sum_k B_k g_T(t - kT)$

$g_T(\cdot) \Rightarrow$ an appropriately chosen finite energy pulse $\left(\int_{-\infty}^{+\infty} g_T^2(t)dt = E < \infty \right)$

M-ary random variable (A_k, B_k) is the Cartesian representation of that symbol in the constellation which represents the k^th m-bit pattern of the coded source data.

- $N(t) \Rightarrow$ broadband zero mean Gaussian noise that is independent of the transmitter output
- The receiver input = $A(t)\cos(w_c t + f) + B(t)\sin(w_c t + f) + N(t)$
- $X_s(t)$ Receiver I-channel output signal component

$X_s(t)$ = $\left(A(t)\cos(w_c t + f) + B(t)\sin(w_c t + f)\right)\cos\left(w_c t + \hat{f}\right) \otimes h(t)$

$$= \left(A(t)\cos(w_c t + f) + B(t)\sin(w_c t + f)\right)\cos\left(w_c t + \hat{f}\right) \otimes h(t)$$

$$= \left(A(t)\cos(w_c t + f)\cos\left(w_c t + \hat{f}\right) + B(t)\sin(w_c t + f)\cos\left(w_c t + \hat{f}\right)\right) \otimes h(t)$$

$$= \left(\frac{1}{2}A(t)\left(\cos\left(2w_c t + f + \hat{f}\right) + \cos\left(f - \hat{f}\right)\right) + \frac{1}{2}B(t)\left(\sin\left(2w_c t + f + \hat{f}\right) + \sin\left(f - \hat{f}\right)\right)\right) \otimes h(t)$$

LTI LPF whose impulse response is h(t) filters the $2w_c$ terms

$$= \frac{1}{2}\left(A(t)\cos\left(f - \hat{f}\right) + B(t)\sin\left(f - \hat{f}\right)\right) \otimes h(t)$$

$$= \frac{1}{2}\left(A(t)\cos(\Delta f) + B(t)\sin(\Delta f)\right) \otimes h(t)$$

where $\Delta f = f - \hat{f}$

$$= \frac{1}{2}A \otimes h(t)\cos(\Delta f) + \frac{1}{2}B \otimes h(t)\sin(\Delta f)$$

$Y_s(t)$ Receiver Q-channel output signal component

$$
\begin{aligned}
Y_s(t) &= \left(A(t)\cos\left(w_c t + f\right) + B(t)\sin\left(w_c t + f\right)\right)\sin\left(w_c t + \hat{f}\right) \otimes h(t) \\
&= \left(A(t)\cos\left(w_c t + f\right)\sin\left(w_c t + \hat{f}\right) + B(t)\sin\left(w_c t + f\right)\sin\left(w_c t + \hat{f}\right)\right) \otimes h(t) \\
&= \left(\frac{1}{2}A(t)\left(\sin\left(2w_c t + f + \hat{f}\right) - \sin\left(f - \hat{f}\right)\right) + \frac{1}{2}B(t)\left(\cos\left(f - \hat{f}\right) - \cos\left(2w_c t + f + \hat{f}\right)\right)\right) \otimes h(t) \\
&= \frac{1}{2}\left(-A(t)\sin\left(f - \hat{f}\right) + B(t)\cos\left(f - \hat{f}\right)\right) \otimes h(t) \\
&= -\frac{1}{2}A \otimes h(t)\sin\left(\Delta f\right) + \frac{1}{2}B \otimes h(t)\cos\left(\Delta f\right)
\end{aligned}
$$

$$
\begin{aligned}
2X_s(t) &= A \otimes h(t)\cos\left(\Delta f\right) + \frac{1}{2}B \otimes h(t)\sin\left(\Delta f\right) \\
&= \left(\sum_k A_k g_T(t - kT)\right) \otimes h(t)\cos\left(\Delta f\right) + \left(\sum_k B_k g_T(t - kT)\right) \otimes h(t)\sin\left(\Delta f\right) \\
&= \left(\sum_k A_k\left(g_T(t - kT) \otimes h(t)\right)\right)\cos\left(\Delta f\right) + \left(\sum_k B_k\left(g_T(t - kT) \otimes h(t)\right)\right)\sin\left(\Delta f\right) \\
&= \left(\sum_k A_k\left(f(t - kT)\right)\right)\cos\left(\Delta f\right) + \left(\sum_k B_k\left(f(t - kT)\right)\right)\sin\left(\Delta f\right) \\
&= \sum_k f(t - kT)\left(A_k\cos\left(\Delta f\right) + B_k\sin\left(\Delta f\right)\right)
\end{aligned}
$$

where $f(t) = \left(g_T \otimes h\right)(t)$

Similarly,

$$
2Y_s(t) = \sum_k f(t - kT)\left(-A_k\sin\left(\Delta f\right) + B_k\cos\left(\Delta f\right)\right)
$$

Introduce the polar representation of the points in the QAM constellation, namely $\left(R_k, \Theta_k\right)$

where $\quad R_k = \sqrt{A^k + B^k} \qquad \Theta_k = \tan^{-1}\left(\frac{B_k}{A_k}\right)$

$\qquad A_k = R_k\cos\left(\Theta_k\right) \qquad B_k = R_k\sin\left(\Theta_k\right)$

$$
\begin{aligned}
2X_s(t) &= \sum_k f(t - kT)\left(R_k\cos\left(\Theta_k\right)\cos\left(\Delta f\right) + R_k\sin\left(\Theta_k\right)\sin\left(\Delta f\right)\right) \\
&= \sum_k R_k\cos\left(\Theta_k - \Delta f\right) f(t - kT) \\
2Y_s(t) &= \sum_k f(t - kT)\left(-R_k\cos\left(\Theta_k\right)\sin\left(\Delta f\right) + R_k\sin\left(\Theta_k\right)\cos\left(\Delta f\right)\right) \\
&= \sum_k R_k\sin\left(\Theta_k - \Delta f\right) f(t - kT)
\end{aligned}
$$

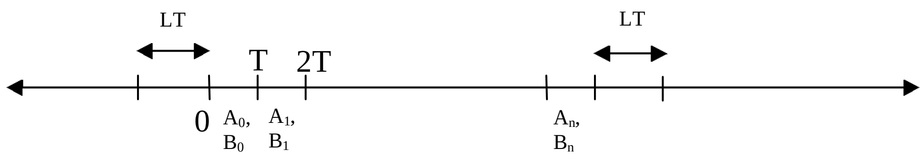
Define $Z_s(t) = X_s(t) + jY_s(t) = \frac{1}{2}\sum_k R_k e^{j\left(\Theta_k - \Delta f\right)} f(t - kT)$

- the overall heterodyne receiver output

$$Z(t) = Z_s(t) + Z_n(t) = \frac{1}{2}\sum_k R_k e^{j(\Theta_k - \Delta f)} f(t - kT) + Z_n(t)$$

- $Z_n(t)$ is
  - $X_n(t)+jY_n(t)$ where $X_n(t)$ and $Y_n(t)$ are the noise components of the I-channel and Q-channel of the receiver, respectively.
  - 0-mean circularly Gaussian random process
  - independent fof the signal
  - distributed in a manner that is independent of the LO phase $\hat{f}$
- To recover data A_0, B_0, A_1, B_1, …, A_n, B_n,

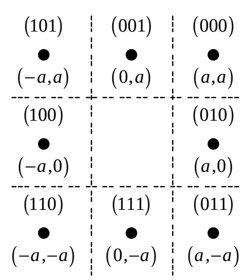
  study Z(t) over some time period, say $-LT \le t \le nT+LT$ where $L \ll n$



Maximum Likelihood Seqence Estimation:
Choose that one of M^{n+1} possible symbols sequences a_0, b_0, …, a_n, b_n
that maximize $P_{\underline{Z}_n}\left(\underline{z} - \underline{z}_s\left(a_0,\ldots,b_n\right)\right)$
(M = constellation size)

QAM example

- 8-ary QAM constellation



where a > 0

- $1 \le i \le 4$

  Message $M_i \rightarrow$ codeword $c_i = \underbrace{x_0x_1x_2}_{\substack{\downarrow \\ (A_1,B_1) \\ A_1+jB_1}}\underbrace{x_3x_4x_5}_{\substack{\downarrow \\ (A_2,B_2) \\ A_2+jB_2}}$

  (A_1, B_1) and (A_2, B_2) are the Cartesian coordinates of the symbols in the QAM constellation that correspond, respectively, to the first three bits and the remaining three bits of the code word for the selected message

  - $P(C = c_i)$ is identical for all i.
- Receiver

- use LPF with h(·) that is designed to eliminate intersymbol interference (ISI)
    - $f(kT-\ell T)=(g_T \otimes h)(kT-\ell T)=d_{k,\ell}=\begin{cases}1 & \text{if } k=\ell \\ 0 & \text{if } k\neq\ell\end{cases}$
- recover the carrier phase perfectly $\Rightarrow$ $\Delta f=0$
- $\underline{Z}_n=\begin{pmatrix}Z_n(T)\\ Z_n(2T)\end{pmatrix}$: characterized by s_1, s_2, r
    - $E\left|Z_n(T)\right|^2=s_1^2$
    - $E\left|Z_n(2T)\right|^2=s_2^2$
    - $EZ_n(T)\overline{Z_n(2T)}=s_1 s_2 r$
    - $P(\underline{Z}_n=\underline{z}_n)=\dfrac{1}{p^2\det\left(K_{\underline{Z}_n}\right)}e^{-\underline{z}_n^{\dagger}K_{\underline{Z}_n}^{-1}\underline{z}_n}$
- From overall output of the receiver $\underline{Z}=\begin{pmatrix}Z(T)\\ Z(2T)\end{pmatrix}=\begin{pmatrix}Z_s(T)+Z_n(T)\\ Z_s(2T)+Z_n(2T)\end{pmatrix}=\begin{pmatrix}z_1\\ z_2\end{pmatrix}$ observed,

  <u>determine M$_i$</u> sent
- To do this,
    - Need to find which *i* maximize $P\left(C=c_i \middle| \underline{Z}=\underline{z}\right)$ for i = 1, 2, 3, 4

      $P\left(C=c_i \middle| \underline{Z}=\underline{z}\right)=\dfrac{P(C=c_i)P\left(\underline{Z}=\underline{z} \middle| C=c_i\right)}{P(\underline{Z}=\underline{z})}$

      $P(C=c_i)$ and $P(\underline{Z}=\underline{z})$ are independent of *i*.

      So, need to find which *i* maximize $P\left(\underline{Z}=\underline{z} \middle| C=c_i\right)$
    - Find $\underline{Z}_S=\underline{z}_{s_i}$ that corresponds to $C=c_i$.

$$C=c_i \rightarrow \begin{pmatrix}(A_1,B_1)\\ (A_2,B_2)\\ \vdots\end{pmatrix} \rightarrow \begin{pmatrix}(a_1,b_1)\\ (a_2,b_2)\\ \vdots\end{pmatrix}_i \rightarrow \begin{pmatrix}a_1+jb_1\\ a_2+jb_2\\ \vdots\end{pmatrix}_i \text{ or } \begin{pmatrix}(r_1,\Theta_1)\\ (r_2,\Theta_2)\\ \vdots\end{pmatrix}_i \rightarrow \begin{pmatrix}Z_s(T)\\ Z_s(2T)\end{pmatrix}_i$$

$$Z_s(t)=\frac{1}{2}\sum_k R_k e^{j(\Theta_k-\Delta f)}f(t-kT)=\frac{1}{2}\sum_k R_k e^{j(\Theta_k-0)}f(t-kT)=\frac{1}{2}\sum_k R_k e^{j(\Theta_k)}f(t-kT)$$

$$\begin{pmatrix} Z_s(T) \\ Z_s(2T) \end{pmatrix}_i = \begin{pmatrix} \sum_{k=1}^{2} R_k e^{j(\Theta_k)} f(T-kT) \\ \sum_{k=1}^{2} R_k e^{j(\Theta_k)} f(2T-kT) \end{pmatrix}_i = \begin{pmatrix} R_1 e^{j(\Theta_1)} f(T-T) + R_2 e^{j(\Theta_2)} f(T-2T) \\ R_1 e^{j(\Theta_1)} f(2T-T) + R_2 e^{j(\Theta_2)} f(2T-2T) \end{pmatrix}_i$$

$$= \begin{pmatrix} R_1 e^{j(\Theta_1)} \cdot 1 + R_2 e^{j(\Theta_2)} \cdot 0 \\ R_1 e^{j(\Theta_1)} \cdot 0 + R_2 e^{j(\Theta_2)} \cdot 1 \end{pmatrix}_i = \begin{pmatrix} R_1 e^{j(\Theta_1)} \\ R_2 e^{j(\Theta_2)} \end{pmatrix}_i = \begin{pmatrix} a_1 + jb_1 \\ a_2 + jb_2 \end{pmatrix}_i$$

- $\underline{Z} = \underline{Z}_s + \underline{Z}_n$. Therefore, $\underline{Z}_n = \underline{Z} - \underline{Z}_s$ and

  $P\left(\underline{Z} = \underline{z} \middle| C = c_i\right) = P\left(\underline{Z} = \underline{z} \middle| \underline{Z}_s = \underline{z}_{s_i}\right) = P\left(\underline{Z}_n = \underline{z} - \underline{z}_{s_i}\right)$

  $P\left(\underline{Z}_n = \underline{z} - \underline{z}_{s_i}\right) = \dfrac{1}{p^2 \det\left(K_{\underline{Z}_n}\right)} e^{-\left(\underline{z}-\underline{z}_{s_i}\right)^{\dagger} K_{\underline{Z}_n}^{-1} \left(\underline{z}-\underline{z}_{s_i}\right)}$

  Let $\underline{z}_{n_i} = \begin{pmatrix} z_{n_{1i}} \\ z_{n_{2i}} \end{pmatrix} = \underline{z} - \underline{z}_{s_i}$

  Find i^* that maximize $P\left(\underline{Z}_n = \underline{z}_{n_i}\right)$

  This i^* minimize $\left(\underline{z}_{n_i}\right)^{\dagger} K_{\underline{Z}_n}^{-1} \left(\underline{z}_{n_i}\right)$

  Since $\left(\left(1-|r|^2\right) s_1 s_2\right) \underline{z}_{n_i}^{\dagger} K_{\underline{Z}_n}^{-1} \underline{z}_{n_i} = \dfrac{s_2}{s_1} \left|z_{n_{1i}}\right|^2 - 2\mathrm{Re}\left\{ r \overline{z_{n_{1i}}} z_{n_{2i}} \right\} + \dfrac{s_1}{s_2} \left|z_{n_{2i}}\right|^2$

  This i^* minimize $\dfrac{s_2}{s_1} \left|z_{n_{1i}}\right|^2 + \dfrac{s_1}{s_2} \left|z_{n_{2i}}\right|^2 - 2\mathrm{Re}\left\{ r \overline{z_{n_{1i}}} z_{n_{2i}} \right\}$

- It is most likely that sender sent M_{i^*}

Moment of Complex (proper) Gaussian vectors

- $E \prod_{a=1}^{A} Z_{k_a} \prod_{b=1}^{B} \overline{Z}_{h_b} = 0$ unless A = B
- For A = B, $E \prod_{a=1}^{A} Z_{k_a} \prod_{b=1}^{B} \overline{Z}_{h_b} = E \prod_{a=1}^{A} Z_{k_a} \overline{Z}_{h_a} = \sum_{P \in P_A} \prod_{a=1}^{A} Z_{k_a} \overline{Z}_{h_{P\{a\}}}$

  where P_A is the set of all permutations P:{1,…,A} → {1,…,A}
- Example: A = 2
  - $E Z_{k_1} Z_{k_2} \overline{Z}_{h_1} \overline{Z}_{h_2} = \left(E Z_{k_1} \overline{Z}_{h_1}\right) \cdot \left(E Z_{k_2} \overline{Z}_{h_2}\right) + \left(E Z_{k_1} \overline{Z}_{h_2}\right) \cdot \left(E Z_{k_2} \overline{Z}_{h_1}\right)$
  - For $k_1 = k_2 = h_1 = h_2$,

    $E|Z|^4 = 2\left(E|Z|^2\right)^2 = 2s_Z^4$