Revision

- $\int_{-\infty}^{+\infty} g(t) d(t-t_0)\, dt = g(t_0)$
- Cauchy-Schwarz inequality:

  $\left|\int fg\right|^2 \le \left(\int |f||g|\right)^2 \le \int |f|^2 \int |g|^2$ with equality if $g = c \times \overline{f}$ where c is a constant

Fourier property

- $h(t) = \int_{-\infty}^{+\infty} H(f) e^{j2p\, ft} df \xrightarrow{FT} H(f) = \int_{-\infty}^{+\infty} h(t) e^{-j2p\, ft} dt$
- $h(-t) \xrightarrow{FT} H(-f) = \overline{H(f)}$

  $$\int_{-\infty}^{+\infty} h(-t) e^{-j2p\, ft} dt = \int_{-\infty}^{+\infty} h(u) e^{j2p\, fu} du = \int_{-\infty}^{+\infty} h(u) e^{-j2p(-f)u} du = H(-f)$$

  $= \overline{H(f)}$ for real *h(t)*
- $h(t-a) \xrightarrow{FT} e^{-j2p\, fa} H(f)$

  $$\int_{-\infty}^{+\infty} h(t-a) e^{-j2p\, ft} dt = \int_{-\infty}^{+\infty} h(u) e^{-j2p\, f(u+a)} du = e^{-j2p\, fa} \int_{-\infty}^{+\infty} h(u) e^{-j2p\, fu} du = e^{-j2p\, fa} H(f)$$
- LTI: $w(t) \otimes h(t) \xrightarrow{FT} W(f) H(f)$

  $$w(t) \otimes h(t) = \int_{-\infty}^{+\infty} w(t) h(t-t)\, dt$$

  $$FT\left(w(t) \otimes h(t)\right) = \int_{-\infty}^{+\infty} w(t) \left(\int_{-\infty}^{+\infty} h(t-t) e^{-j2p\, ft} dt \right) dt$$

  $$= \int_{-\infty}^{+\infty} w(t) \left(H(f) e^{-j2p\, ft}\right) dt = H(f) \int_{-\infty}^{+\infty} w(t) e^{-j2p\, ft} dt$$

  $$= W(f) H(f)$$
- $h(t) e^{+j2p\, f_0 t} \xrightarrow{FT} H(f - f_0)$

  $$\int_{-\infty}^{+\infty} \left(h(t) e^{+j2p\, f_0 t}\right) e^{-j2p\, ft} dt = \int_{-\infty}^{+\infty} h(t) e^{-j2p(f-f_0)t} dt = H(f - f_0)$$
- Parseval's Relation: $\int_{-\infty}^{+\infty} |w(t)|^2 dt = \int_{-\infty}^{+\infty} |W(f)|^2 df$

$$\int_{-\infty}^{+\infty} |w(t)|^2 \, dt = \int_{-\infty}^{+\infty} w(t)\overline{w(t)}dt = \int_{-\infty}^{+\infty} \overline{w(t)} \int_{-\infty}^{+\infty} W(f) e^{j2p\,ft} df\,dt$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \overline{w(t)} W(f) e^{j2p\,ft} df\,dt$$

$$\int_{-\infty}^{+\infty} |W(f)|^2 \, df = \int_{-\infty}^{+\infty} W(f)\overline{W(f)}df = \int_{-\infty}^{+\infty} W(f) \overline{\int_{-\infty}^{+\infty} W(t) e^{-j2p\,ft} dtdf}$$

$$= \int_{-\infty}^{+\infty} W(f) \int_{-\infty}^{+\infty} \overline{W(t)} e^{j2p\,ft} dtdf = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} W(f)\overline{W(t)} e^{j2p\,ft} df\,dt$$

- Duality: $g(t) = H(t) \xrightarrow{FT} G(f) = h(f)$

$$g(t) = H(t) = \int_{-\infty}^{+\infty} h(u) e^{-j2p\,tu} du$$

$$G(f) = \int_{-\infty}^{+\infty} g(t) e^{-j2p\,ft} dt = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(u) e^{-j2p\,tu} du e^{-j2p\,ft} dt = \int_{-\infty}^{+\infty} h(u) \underbrace{\left(\int_{-\infty}^{+\infty} e^{-j2p\,tu} e^{-j2p\,ft} dt \right)}_{FT\left(e^{-j2p\,tu}\right)} du$$

$$= \int_{-\infty}^{+\infty} h(u) d(f-u) du = h(f)$$

• $d(t) \xrightarrow{FT} 1$ $\int_{-\infty}^{+\infty} d(t) e^{-j2p\,ft} dt = e^{-j2p\,t0} = 1$	• $1 \xrightarrow{FT} d(f)$ $\int_{-\infty}^{+\infty} d(f) e^{j2p\,ft} df = e^{j2p\,0t} = 1$

- $1 \cdot e^{j2p\,f_0 t} \xrightarrow{FT} d(f - f_0)$
- $a(t) = \begin{cases} a e^{-at} & t \geq 0 \\ 0 & t < 0 \end{cases} \xrightarrow{FT} \dfrac{a}{a + j2p\,f}$

$$A(f) = \int_{-\infty}^{+\infty} a(t) e^{-j2p\,ft} dt = \int_{0}^{+\infty} a e^{-at} e^{-j2p\,ft} dt = a \int_{0}^{+\infty} e^{-(a+j2p\,f)t} dt$$

$$= -\frac{a}{a + j2p\,f} e^{-(a+j2p\,f)t} \Big|_0^{\infty} = \frac{a}{a + j2p\,f}$$

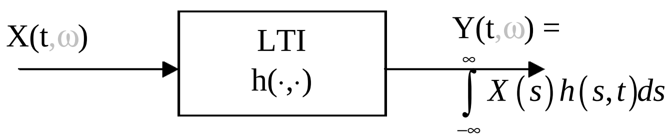
- $\dfrac{2k}{k^2 + (2pt)^2} \xrightarrow{FT} e^{-k|f|}$

$$\int_{-\infty}^{+\infty} e^{-k|f|} e^{j2p\,ft} df = \int_{-\infty}^{0} e^{kf} e^{j2p\,ft} df + \int_{0}^{+\infty} e^{-kf} e^{j2p\,ft} df$$

$$= \int_{-\infty}^{0} e^{(k+j2pt)f} df + \int_{0}^{+\infty} e^{(-k+j2pt)f} df = \frac{1}{k+j2pt} + \frac{1}{k-j2pt} = \frac{2k}{k^2+(2pt)^2}$$

Filtering

Linear time-varying filter



- $Y(t) = \int_{-\infty}^{\infty} X(s)h(s,t)ds$

- $m_Y(t) = \int_{-\infty}^{\infty} m_X(s)h(s,t)ds$

$$m_Y(t) = EY(t) = E\left[\int_{-\infty}^{\infty} X(s)h(s,t)ds\right] = \int_{-\infty}^{\infty} E[X(s)]h(s,t)ds = \int_{-\infty}^{\infty} m_X(s)h(s,t)ds$$

- $R_{YY}(s,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u,s)h(v,t)R_{XX}(s,t)dudv$

$$Y(s) = \int_{-\infty}^{\infty} X(u)h(u,s)du \ ,\ Y(t) = \int_{-\infty}^{\infty} X(v)h(v,t)dv$$

$$R_{YY}(s,t) = EY(s)Y(t) = E\int_{-\infty}^{\infty} X(u)h(u,s)du \int_{-\infty}^{\infty} X(v)h(v,t)dv$$

$$= E\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u,s)h(v,t)X(u)X(v)dudv = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u,s)h(v,t)EX(u)X(v)dudv$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u,s)h(v,t)R_{XX}(s,t)dudv$$

LTI

- $h(s,t) = h(t-s)$

- $Y(t) = \int_{-\infty}^{+\infty} h(t)X(t-t)dt = \int_{-\infty}^{+\infty} h(t-t)X(t)dt = X \otimes h$

- $m_Y(t) = \int\limits_{-\infty}^{\infty} m_X(s)h(t-s)\,ds = m_X \otimes h$

- ***$R_h(t)$* System correlation function** : $R_h(t) = \int\limits_{-\infty}^{+\infty} h(t')h(t'-t)dt' = h \otimes g$

  where $g(t) = h(-t)$

$$h(t) \otimes h(-t) = \int\limits_{-\infty}^{+\infty} h(t)h(-(t-t))dt$$

- $R_h(-t) = R_h(t)$

$$R_h(-t) = \int\limits_{-\infty}^{+\infty} h(t')h(t'+t)dt'$$

$$t'' = t' + t \Rightarrow dt'' = dt',\ t' = t'' - t$$

$$R_h(-t) = \int\limits_{-\infty}^{+\infty} h(t''-t)h(t'')dt'' = R_h(t)$$

LTI and w.s.s. X(t)

- If the input to a time-invariant, stable linear system is a wide-sense stationary random process

  then the output of that system is also a wide-sense stationary random process

$$\text{w.s.s. } X(t) \longrightarrow \boxed{\begin{array}{c}\text{LTI}\\ h(\cdot)\end{array}} \longrightarrow \text{w.s.s. } Y(t) = \int\limits_{-\infty}^{+\infty} h(t-t)X(t)dt$$

- $m_Y(t) = E\left[Y(t)\right] = m_X \int\limits_{-\infty}^{+\infty} h(s)ds$ constant = m_Y

- $m_Y(t) = E\left[Y(t)\right] = \int\limits_{-\infty}^{\infty} m_X(s)h(s,t)\,ds = m_X \int\limits_{-\infty}^{+\infty} h(t-s)ds = m_X \int\limits_{-\infty}^{+\infty} h(w)dw$

- $R_Y(t) = \int\limits_{-\infty}^{+\infty} R_X(t-w)R_h(w)dw = R_X \otimes R_h$

$$R_Y(t_1,t_2) = EY_{t_1}Y_{t_2} = E\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(t_1)h(t_2)X(t_1-t_1)X(t_2-t_2)dt_1dt_2$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(t_1)h(t_2)E\left[X(t_1-t_1)X(t_2-t_2)\right]dt_1dt_2$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(t_1)h(t_2)R_X\left((t_1-t_1)-(t_2-t_2)\right)dt_1dt_2$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(t_1)h(t_2)R_X\left((t_1-t_2)-(t_1-t_2)\right)dt_1dt_2$$

$$R_Y(t) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(u)h(v)R_X\left(t-(u-v)\right)dudv$$

In v-integral, substitute with $v = u\text{-}w$, $dv = \text{-}dw$

$$R_Y(t) = \int_{-\infty}^{+\infty} h(u)\left(\int_{-\infty}^{+\infty} h(v)R_X\left(t-(u-v)\right)dv\right)du$$

$$= \int_{-\infty}^{+\infty} h(u)\left(-\int_{+\infty}^{-\infty} h(u-w)R_X\left(t-(u-(u-w))\right)dw\right)du$$

$$= \int_{-\infty}^{+\infty} h(u)\left(\int_{-\infty}^{+\infty} h(u-w)R_X(t-w)dw\right)du$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(u-w)h(u)R_X(t-w)dwdu$$

$$= \int_{-\infty}^{+\infty} R_X(t-w)\left(\int_{-\infty}^{+\infty} h(u-w)h(u)du\right)dw$$

Use $R_h(w) = \int_{-\infty}^{+\infty} h(u)h(u-w)du$

$$R_Y(t) = \int_{-\infty}^{+\infty} R_X(t-w)R_h(w)dw = R_X \otimes R_h$$

- $K_Y(t_1,t_2) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(t_1)h(t_2)K_X\left((t_1-t_2)-(t_1-t_2)\right)dt_1dt_2$

  $K_Y(t) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(t_1)h(t_2)K_X\left(t-(t_1-t_2)\right)dt_1dt_2$

$$K_Y(t_1,t_2)=R_Y(t_1,t_2)-EY_{t_1}EY_{t_2}$$

$$=\left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}h(t_1)h(t_2)E\left[X(t_1-t_1)X(t_2-t_2)\right]dt_1dt_2\right)$$

$$-\left(\int_{-\infty}^{+\infty}h(t_1)E\left[X(t_1-t_1)\right]dt_1\right)\left(\int_{-\infty}^{+\infty}h(t_2)E\left[X(t_2-t_2)\right]dt_2\right)$$

$$=\left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}h(t_1)h(t_2)E\left[X(t_1-t_1)X(t_2-t_2)\right]dt_1dt_2\right)$$

$$-\left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}h(t_1)h(t_2)E\left[X(t_1-t_1)\right]E\left[X(t_2-t_2)\right]dt_1dt_2\right)$$

$$=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}h(t_1)h(t_2)\begin{pmatrix}E\left[X(t_1-t_1)X(t_2-t_2)\right]\\ -E\left[X(t_1-t_1)\right]E\left[X(t_2-t_2)\right]\end{pmatrix}dt_1dt_2$$

$$=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}h(t_1)h(t_2)K_X(t_1-t_1,t_2-t_2)dt_1dt_2$$

$$=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}h(t_1)h(t_2)K_X((t_1-t_1)-(t_2-t_2))dt_1dt_2$$

$$=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}h(t_1)h(t_2)K_X((t_1-t_2)-(t_1-t_2))dt_1dt_2$$

- $R_{YX}(t)=\int_{-\infty}^{+\infty}h(t')R_X(t-t')dt'$

$$R_{YX}(t)=E[Y_{t+t}X_t]=E\left[\left(\int_{-\infty}^{+\infty}h(t')X(t+t-t')dt'\right)X(t)\right]$$

$$=\int_{-\infty}^{+\infty}h(t')E\left[X(t+t-t')X(t)\right]dt'$$

$$=\int_{-\infty}^{+\infty}h(t')R_X(t+t-t',t)dt'=\int_{-\infty}^{+\infty}h(t')R_X(t+t-t'-t)dt'$$

$$=\int_{-\infty}^{+\infty}h(t')R_X(t-t')dt'$$

- $\boxed{R_Y(t)=\int_{-\infty}^{+\infty}R_h(t')R_X(t-t')dt'=R_h(t)\otimes R_X(t)}$

$$R_Y(t)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}h(t_1)R_X(t-(t_1-t_2))h(t_2)dt_2dt_1$$

$$t'=t_1-t_2\Rightarrow t_2=t_1-t'\Rightarrow dt_2=-dt'$$

$$R_Y(t) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(t_1)h(t_1 - t')R_X(t - t')dt'dt_1$$

$$= \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} h(t_1)h(t_1 - t')dt_1\right)R_X(t - t')dt'$$

$$= \int_{-\infty}^{+\infty} R_h(t')R_X(t - t')dt' = R_h(t) \otimes R_X(t)$$

- $VAR(Y_t) = \int_{-\infty}^{+\infty} R_h(t')K_Y(t')dt'$

$$VAR(Y_t) = s_{Y_t}^{\ 2} = K_Y(0) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(t_1)h(t_2)K_Y(0-(t_1 - t_2))dt_1dt_2$$

$$t' = t_2 - t_1 \Rightarrow t_1 = t_2 - t' \Rightarrow dt_1 = -dt'$$

$$VAR(Y_t) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(t_2 - t')h(t_2)K_Y(t')dt'dt_2$$

$$= \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} h(t_2)h(t_2 - t')dt_2\right)K_Y(t')dt'$$

$$= \int_{-\infty}^{+\infty} R_h(t')K_Y(t')dt'$$

Power Spectral Density

- (Power) spectral density (p.s.d.) of w.s.s. $\{X(t)\}$:

  $S_X(f)$ = $\int_{-\infty}^{+\infty} R_X(t)e^{-j2p\,ft}dt$ = FT of R_X

- $R_X(t) = \int_{-\infty}^{+\infty} S_X(f)e^{j2p\,ft}df \xrightarrow{FT} S_X(f) = \int_{-\infty}^{+\infty} R_X(t)e^{-j2p\,ft}dt$

- $S_X(f) = S_X(-f)$

$$S_X(-f) = \int_{-\infty}^{+\infty} R_X(t)e^{-j2p(-f)t}dt = \int_{-\infty}^{+\infty} R_X(t)e^{-j2p\,f(-t)}dt$$

$$= -\int_{+\infty}^{-\infty} R_X(-u)e^{-j2p\,fu}du$$

$$= \int_{-\infty}^{+\infty} R_X(u)e^{-j2p\,fu}du \quad ;R_X(-u) = R_X(u)$$

$$= S_X(f)$$

- **spectral analysis**: $S_Y(f) = S_X(f)\left|H(f)\right|^2$

$$R_Y(t) = R_h(t) \otimes R_X(t)$$

$$S_Y(f) = FT\left(R_h(t)\right)S_X(f) = FT(h \otimes g)S_X(f) = H(f)G(f)S_X(f)$$

$$g(t) = h(-t) \xrightarrow{FT} G(f) = H(-f) = \overline{H(f)}$$

- The **average power** in {*X(t)*} over 0 ≤ t ≤ T is $E\frac{1}{T}\int_0^T X^2(t)dt = R_X(0) = \int_{-\infty}^{\infty} S_X(f)df$

$$E\frac{1}{T}\int_0^T X^2(t)dt = \frac{1}{T}\int_0^T EX^2(t)dt$$

$$= \frac{1}{T}\int_0^T R_X(0)dt \text{ if } \{X(t)\} \text{ is w.s.s.}$$

$$= R_X(0) \text{ ; not depend on T}$$

$$= \int_{-\infty}^{\infty} S_X(f)e^{2p f 0}df = \int_{-\infty}^{\infty} S_X(f)df \quad ; R_X(t) = \int_{-\infty}^{\infty} S_X(f)e^{2p f t}df$$

- $S_X(f)$ measures the average power in {*x(t)*} per Hz in vicinity of *f*.
- $\int_A S_X(f)df$ equals the average power of the random “fluctuations” of {X(t)} in the frequency band A (and may be its negative image)

- $S_X(f) \geq 0$

Because $R_X(t)$ is a non-negative definite function, and

(Bochner-Khinchin Theorem) every nnd. function has a non-negative FT.

- White noise
  - $S_N(f) = N_0$
  - $R_N(t) = N_0 d(t)$

$$R_N(t) = \int_{-\infty}^{+\infty} N_0 e^{j2p f t}df = N_0 d(t)$$

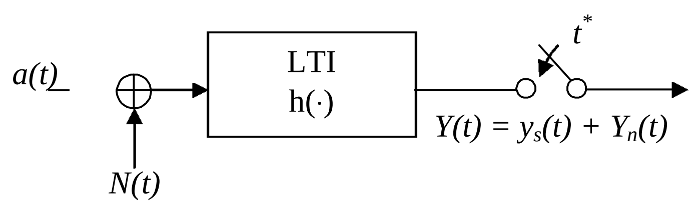
Discrete case

- $A(f) = \sum_{k=-\infty}^{\infty} a[k]e^{-j2p fk}$; $|f| \leq \frac{1}{2}$, periodic with period 1

$$a[k] = \int_{-\frac{1}{2}}^{\frac{1}{2}} A(f) e^{-j2\pi fk} df$$

- $S_A(f) = \sum_{k=-\infty}^{\infty} R_A[k] e^{-j2\pi fk}$; $|f| \le \frac{1}{2}$, periodic with period 1
- Ex. i.i.d. process $\{X_k\}$, each X_k having mean μ and variance σ^2
  - $R_X[k] = EX_n X_{n-k} = \begin{cases} EX_n^2 = \mu^2 + \sigma^2 & \text{if } k = 0 \\ EX_n EX_{n-k} = \mu^2 & \text{if } k \ne 0 \end{cases} = \mu^2 + \sigma^2 \delta[k]$
  - $S_X(f) = \sum_{k=-\infty}^{\infty} R_X[k] e^{-j2\pi fk} = \sum_{k=-\infty}^{\infty} \mu^2 e^{-j2\pi fk} + \sum_{k=-\infty}^{\infty} \sigma^2 \delta[k] e^{-j2\pi fk} = \mu^2 \sum_{k=-\infty}^{\infty} e^{-j2\pi fk} + \sigma^2$
    $= \sigma^2 + \mu^2 \delta(f)$;periodic with period = 1

Matched Filtering



$\{N(t)\}$ is a w.s.s.,
0-mean r.p.

- input : a(t) + N(t)
  - deterministic signal $a(t)$ has finite energy $E = \int_{-\infty}^{\infty} a^2(t) dt$ = energy of $a(t)$
  - $\{N(t)\}$ is a w.s.s., 0-mean r.p.
- output : $Y(t) = y_s(t) + Y_n(t)$
  - $y_s(t) = a \otimes h(t) = \int_{-\infty}^{\infty} a(\tau) h(t-\tau) d\tau = \int_{-\infty}^{\infty} A(f) H(f) e^{j2\pi ft} df$
  - $Y_n(t) = N(t) \otimes h(t) = \int_{-\infty}^{\infty} N(\tau) h(t-\tau) d\tau$
    - $Y_n(t)$ is w.s.s. since $N(t)$ is w.s.s.

- Output signal power at t is $y_s^2(t) = \left(\int_{-\infty}^{\infty} A(f) H(f) e^{j2\pi ft} df \right)^2 = \left(\int_{-\infty}^{\infty} a(\tau) h(t-\tau) d\tau \right)^2$

$$Y_s(f) = A(f) H(f)$$

$$y_s(t)=\int_{-\infty}^{\infty} Y_s(f)e^{j2\pi ft}df=\int_{-\infty}^{\infty} A(f)H(f)e^{j2\pi ft}df$$

$$y_s^2(t)=\left(\int_{-\infty}^{\infty} A(f)H(f)e^{j2\pi ft}df\right)^2$$

- Output average noise power is $EY_n^2(t)=R_{Y_n}(0)=\int_{-\infty}^{\infty} S_N(f)\left|H(f)\right|^2 df$

  for any time ($Y_n(t)$ is w.s.s.)

$$S_{Y_n}(f)=S_N(f)\left|H(f)\right|^2$$

$$R_{Y_n}(0)=\int_{-\infty}^{\infty} S_{Y_n}(f)df=\int_{-\infty}^{\infty} S_N(f)\left|H(f)\right|^2 df$$

- $$SNR_{out}(t^*)=\frac{y_s^2(t^*)}{EY_n^2(t^*)}=\frac{y_s^2(t^*)}{R_{Y_n}(0)}=\frac{\left(\int_{-\infty}^{\infty} A(f)H(f)e^{j2\pi ft^*}df\right)^2}{\int_{-\infty}^{\infty} S_N(f)\left|H(f)\right|^2 df}=\frac{\left(\int_{-\infty}^{\infty} a(\tau)h(t-\tau)d\tau\right)^2}{\int_{-\infty}^{\infty} S_N(f)\left|H(f)\right|^2 df}$$

- Object : Design h(·) so that $Y(t^*)$ has maximum SNR

  Matched filter:

  Only $H(f)=c\times\frac{\overline{A(f)}}{S_N(f)}e^{-j2\pi ft^*}$ gives maximized $SNR_{out}(t^*)=\int_{-\infty}^{\infty}\frac{\left|A(f)\right|^2}{S_N(f)}df$

  where c is some constant

$$SNR_{out}(t^*)=\frac{y_s^2(t^*)}{EY_n^2(t^*)}=\frac{\left(\int_{-\infty}^{\infty} A(f)H(f)e^{j2\pi ft^*}df\right)^2}{\int_{-\infty}^{\infty} S_N(f)\left|H(f)\right|^2 df}$$

$$=\frac{\left(\int_{-\infty}^{\infty} A(f)H(f)e^{j2\pi ft^*}df\right)^2}{\int_{-\infty}^{\infty} S_N(f)\left|H(f)\right|^2 df}$$

$$=\frac{\left(\int_{-\infty}^{\infty}\left(\frac{A(f)}{\sqrt{S_N(f)}}e^{j2\pi ft^*}\right)\left(\sqrt{S_N(f)}H(f)\right)df\right)^2}{\int_{-\infty}^{\infty} S_N(f)\left|H(f)\right|^2 df}$$

$y_s^2\left(t^*\right)$ is real, thus $y_s^2\left(t^*\right)=\left|y_s\left(t^*\right)\right|^2$

$$SNR_{out}\left(t^*\right)=\frac{\left|\int\limits_{-\infty}^{\infty}\left(\frac{A(f)}{\sqrt{S_N(f)}}e^{j2p\,ft^*}\right)\left(\sqrt{S_N(f)}H(f)\right)df\right|^2}{\int\limits_{-\infty}^{\infty}S_N(f)\left|H(f)\right|^2df}$$

Cauchy-Schwarz inequality $\Rightarrow$

$$SNR_{out}\left(t^*\right)\le\frac{\left(\int\limits_{-\infty}^{\infty}\left|\left(\frac{A(f)}{\sqrt{S_N(f)}}\right)\right|^2df\right)\cancel{\left(\int\limits_{-\infty}^{\infty}\left|\left(\sqrt{S_N(f)}H(f)e^{j2p\,ft^*}\right)\right|^2df\right)}}{\cancel{\int\limits_{-\infty}^{\infty}S_N(f)\left|H(f)\right|^2df}}\ ;\ \left|e^{jx}\right|=1$$

Thus $SNR_{out}\left(t^*\right)\le\int\limits_{-\infty}^{\infty}\left|\left(\frac{A(f)}{\sqrt{S_N(f)}}e^{j2p\,ft^*}\right)\right|^2df=\int\limits_{-\infty}^{\infty}\frac{\left|A(f)\right|^2}{S_N(f)}df$.

max $SNR_{out}\left(t^*\right)=\int\limits_{-\infty}^{\infty}\frac{\left|A(f)\right|^2}{S_N(f)}df$

if $\sqrt{S_N(f)}H(f)e^{j2p\,ft^*}=c\times\overline{\frac{A(f)}{\sqrt{S_N(f)}}}$

$$\Rightarrow\ H(f)=c\times\frac{\overline{A(f)}}{S_N(f)}e^{-j2p\,ft^*}$$

- Special case: white noise $S_N(f)=N_0,\ R_N(t)=N_0 d(t)$

- $h(t)=a\left(-\left(t-t^*\right)\right)=a\left(t^*-t\right)\xrightarrow{FT}H(f)=c'\times\overline{A(f)}e^{-j2p\,ft^*}$

Then $H(f)=c\times\frac{\overline{A(f)}}{S_N(f)}e^{-j2p\,ft^*}=c\times\frac{\overline{A(f)}}{N_0}e^{-j2p\,ft^*}=c'\times\overline{A(f)}e^{-j2p\,ft^*}$

$a(-t)\xrightarrow{FT}\overline{A(f)}$

$a\left(-\left(t-t^*\right)\right)=a\left(t^*-t\right)\xrightarrow{FT}\overline{A(f)}e^{-j2p\,ft^*}$

- max $SNR_{out}\left(t^*\right)=\frac{E}{N_0}$

$$\max SNR_{out}\left(t^*\right) = \int_{-\infty}^{\infty} \frac{\left|A(f)\right|^2}{S_N(f)} df = \int_{-\infty}^{\infty} \frac{\left|A(f)\right|^2}{N_0} df$$

$$= \frac{1}{N_0} \int_{-\infty}^{\infty} \left|A(f)\right|^2 df \underset{parseval}{=} \frac{1}{N_0} \int_{-\infty}^{\infty} a^2(t)\, df = \frac{E}{N_0}$$

- $y_s\left(t^*\right) = E$

$$h(t) = a\left(-\left(t - t^*\right)\right)$$

$$h(t - t) = a\left(-\left(t - t - t^*\right)\right)$$

$$y_s(t) = \int_{-\infty}^{\infty} a(t)\, h(t - t)\, dt = \int_{-\infty}^{\infty} a(t)\, a\left(-\left(t - t - t^*\right)\right) dt$$

$$y_s\left(t^*\right) = \int_{-\infty}^{\infty} a(t)\, a\left(-\left(\cancel{t^*} - t - \cancel{t^*}\right)\right) dt = \int_{-\infty}^{\infty} a(t)\, a(t)\, dt = E$$

- $R_{Y_n}(0) = EN_0$

$$R_{Y_n}(0) = \int_{-\infty}^{\infty} S_{Y_n}(f) df = \int_{-\infty}^{\infty} S_N(f) \left|H(f)\right|^2 df = N_0 \int_{-\infty}^{\infty} \left|H(f)\right|^2 df = N_0 \int_{-\infty}^{\infty} \left|H(f)\right|^2 df$$

$$= N_0 \int_{-\infty}^{\infty} h^2(t) dt = N_0 \int_{-\infty}^{\infty} a^2\left(-\left(t - t^*\right)\right) dt = EN_0$$

Again, $\max SNR_{out}\left(t^*\right) = \dfrac{y_s^2\left(t^*\right)}{R_{Y_n}(0)} = \dfrac{E^2}{N_0 E} = \dfrac{E}{N_0}$