# Math

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{i=0}^{\infty} \frac{x^i}{i!}$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots = \sum_{i=0}^{\infty} \frac{(-x)^i}{i!}$$

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots = \sum_{i=0}^{\infty} \frac{(x)^{2i}}{(2i)!}$$

- $f(t_1,t_2) = f(t_1+\delta, t_2+\delta)\ \forall\delta \rightarrow f(t_1-t_2)$
- Addition formulas

  $$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

  $$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

  $$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

  $$\cot(A \pm B) = \frac{\cot A \cot B \mp 1}{\cot B \pm \cot A}$$

- Sum, difference and product of trigonometry function

  $$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A+B}{2}\right)$$

  $$\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

  $$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

  $$\cos A - \cos B = 2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

  $$\sin A \sin B = \frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right)$$

  $$\cos A \cos B = \frac{1}{2}\left(\cos(A-B) + \cos(A+B)\right)$$

  $$\sin A \cos B = \frac{1}{2}\left(\sin(A-B) - \sin(A+B)\right)$$

# Revision

- Cauchy-Schwarz Inequality : $(EXY)^2 \le EX^2EY^2$

  Proof:

  Let $Q(l) = E(X - lY)^2 = EX^2 - 2l\,EXY + l^2EY^2 \geq 0$

Let min Q(λ) occurs at $\lambda_0$, then

$$\frac{d}{d\,l}Q(l) = -2EXY + 2\,l\,EY^2 = 0 \Rightarrow l_0 = \frac{EXY}{EY^2}$$

$$Q(l_0) = EX^2 - 2\frac{EXY}{EY^2}EXY + \left(\frac{EXY}{EY^2}\right)^2 EY^2 = \frac{1}{EY^2}\left(EX^2EY^2 - (EXY)^2\right)$$

If $Q(\lambda_0) > 0$, then $\frac{1}{EY^2}\left(EX^2EY^2 - (EXY)^2\right) > 0 \Rightarrow EX^2EY^2 > (EXY)^2$

If $Q(\lambda_0) = 0$, then $\frac{1}{EY^2}\left(EX^2EY^2 - (EXY)^2\right) = 0 \Rightarrow EX^2EY^2 = (EXY)^2$

- X and Y are uncorrelated iff EXY = EX EY

- Y = g(X) → $f_{Y_t}(y) = \left|\frac{d}{dy}g^{-1}(y)\right| f_{X_t}\left(g^{-1}(y)\right)$

## Random Process

- $X(t,\omega) \triangleq \{X_t,\ t \in T\}$
  - If we fix the value of t, say as t′, then we get random variable $X_{t'}(s) = X(t', s)$
  - If we fix the value of s, say as s′, then we get the **sample function** $X_{s'}(t) = X(t, s')$

## Covariance and Correlation Function

- **autocorrelation function** of the random process $\{X_t\}$

  $R_X(t_1,t_2) = E\left[X_{t_1}X_{t_2}\right]$
- cross-correlation function

  $R_{XY}(t_1,t_2) = E\left[X_{t_1}Y_{t_2}\right]$
- **covariance function** of the random process $\{X_t\}$

  $K_X(t_1,t_2) = \operatorname{cov}\left(X_{t_1}, X_{t_2}\right)$
  - $= E\left(\left(X_{t_1} - EX_{t_1}\right)\left(X_{t_2} - EX_{t_2}\right)\right) = EX_{t_1}X_{t_2} - EX_{t_1}EX_{t_2}$
  - $= R_X(t_1,t_2) - E\left[X_{t_1}\right]E\left[X_{t_2}\right]$
- $K_{XY}(t_1,t_2) = \operatorname{cov}\left(X_{t_1}, Y_{t_2}\right)$
  - $= R_{XY}(t_1,t_2) - E\left[X_{t_1}\right]E\left[Y_{t_2}\right]$

Example

- Xi ~ bernoulli(p) , i.i.d.

- EX = p(1) + (1-p)(0) = p
- $EX^2 = p(1)^2 + (1-p)0^2 = p$
- VAR(X) =
  - $E(X-EX)^2 = p(1-p)^2 + (1-p)(0-p)^2 = (1-p)(p)(1-p+p) = p(1-p)$
  - $EX^2 - (EX)^2 = p - p^2 = p(1-p)$

- $Y_j = \sum_{i=1}^{j} X_i$
  - $Y_i \sim$ binomial(i,p)
  - $EY_n = nEX = np$
  - $EY_n^{\,2} = E\left(\sum_{i=1}^{n} X_i\right)^2 = nEX^2 + \underbrace{\sum_{i \neq j} EX_i X_j}_{n^2 - n \text{ terms}} = nEX^2 + \left(n^2 - n\right)\left(EX\right)^2$

    $= np + \left(n^2 - n\right) p^2 = np + n^2 p^2 - np^2$
  - VAR($Y_n$) = $E\left(Y_n - EY_n\right)^2$ =
    - n VAR(X) = np(1-p)
    - $EY_n^{\,2} - \left(EY_n\right)^2 = \left(np + n^2 p^2 - np^2\right) - \left(np\right)^2 = np - np^2 = np\left(1-p\right)$
  - VAR($Y_m$-$Y_n$) where $m \neq n$
    - Observe:
      - If m > n,

        $Y_m - Y_n = \sum_{i=n+1}^{m} X_i$
      - If m < n, consider $Y_n$-$Y_m$ instead (since VAR(-Z) = VAR(Z))

        $Y_n - Y_m = \sum_{i=m+1}^{n} X_i$
    - In any case, this is the sum of |m-n| different $X_i$, so
      VAR($Y_m$-$Y_n$) = |m – n| p(1-p)
  - cov($Y_m$,$Y_n$) where $m \neq n$

    $VAR\left(Y_m - Y_n\right) = E\left(Y_m - Y_n\right)^2 - \left(E\left(Y_m - Y_n\right)\right)^2$

    $= E\left(Y_m - Y_n\right)^2 - \left(EY_m - EY_n\right)^2$

    $= \left(EY_m^{\,2} + EY_n^{\,2} - 2EY_m Y_n\right) - \left(\left(EY_m\right)^2 + \left(EY_n\right)^2 - 2EY_m EY_n\right)$

    $= \left(EY_m^{\,2} - \left(EY_m\right)^2\right) + \left(EY_n^{\,2} - \left(EY_n\right)^2\right) - 2\left(EY_m Y_n - EY_m EY_n\right)$

    $= VAR\left(Y_m\right) + VAR\left(Y_n\right) - 2\operatorname{cov}\left(Y_m, Y_n\right)$

$$\operatorname{cov}(Y_m, Y_n)$$
$$= \frac{1}{2}\left(VAR(Y_m) + VAR(Y_n) - VAR(Y_m - Y_n)\right)$$
$$= \frac{1}{2}\left(mp(1-p) + np(1-p) - |m-n|\,p(1-p)\right) = \frac{1}{2}p(1-p)\left(m+n-|m-n|\right)$$
$$= \begin{cases} \frac{1}{2}p(1-p)(m+n-m+n) & ;\text{ if m > n} \\ \frac{1}{2}p(1-p)(m+n-n+m) & ;\text{ if m < n} \end{cases} = \begin{cases} np(1-p) & ;\text{ if m > n} \\ mp(1-p) & ;\text{ if m < n} \end{cases}$$
$$= p(1-p)\min(m,n)$$

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Example : **Random-telegraph process**

- Def:
  - Real random process $\{X_t, -\infty < t < +\infty\}$
  - $P\{X_t = 0] = P[X_t = 1] = \frac{1}{2}\ \forall t$
  - $P(k,\tau) = \dfrac{(l t)^k e^{-l t}}{k!}$ for k = 0,1, 2, … ~ Poisson

    = probability that k transversals from one value to another occur in a time interval of length $\tau$
    - $\lambda$ = average number of transversals per unit time
  - Occurrence of k transversals in an interval of length $\tau$ is statistically independent of the value assumed by any particular sample function at the start of the given interval
- $P\left(X_{t_1} = X_{t_2}\right) = P\left(X_{t_1} = 1 \mid X_{t_2} = 1\right) = P\left(X_{t_1} = 0 \mid X_{t_2} = 0\right) \sum_{even\, k} P\left(k, |t_1 - t_2|\right)$

  $= \frac{1}{2}\left(1 + e^{-2 l t}\right)$
- $EX_t = 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{1}{2}$
- $R_X(t_1, t_2) = EX_{t_1}X_{t_2}$

  $= 1 \cdot 1 \cdot P\left(X_{t_1} = 1, X_{t_2} = 1\right) + 1 \cdot 0 \cdot P\left(X_{t_1} = 1, X_{t_2} = 0\right)$

  $\quad + 0 \cdot 1 \cdot P\left(X_{t_1} = 0, X_{t_2} = 1\right) + 0 \cdot 0 \cdot P\left(X_{t_1} = 0, X_{t_2} = 0\right)$

  $= P\left(X_{t_1} = 1, X_{t_2} = 1\right)$

$$= P\left(X_{t_1} = 1 \mid X_{t_2} = 1\right) P\left(X_{t_2} = 1\right) = \frac{1}{2} P\left(X_{t_1} = 1 \mid X_{t_2} = 1\right)$$

$$= \frac{1}{2} \sum_{even\, k} P\left(k, \left|t_1 - t_2\right|\right) = \frac{1}{2}\left(e^{-lt} \frac{e^{lt} + e^{-lt}}{2}\right) = \frac{1}{4}\left(1 + e^{-2lt}\right)$$

$$= \frac{1}{4}\left(1 + e^{-2l\left|t_1 - t_2\right|}\right)$$

- $K_X\left(t_1, t_2\right) = R_X\left(t_1, t_2\right) - EX_{t_1} EX_{t_2} = R_X\left(t_1, t_2\right) - \frac{1}{4} = \frac{1}{4} e^{-2l\left|t_1 - t_2\right|}$

## Stationary

- The random process $\{X_t, t \in T\}$ is **(strictly) stationary** if and only if

  all of the finite-dimensional probability distribution functions are invariant under shifts of the time origin.

  That is

  $\forall n$ (including $n = 1$), and

  $\forall$set of time instants $\{t_i \in T, i = 1, 2, \ldots, n\}$

  $F_{X_{t_1}, X_{t_2}, \ldots, X_{t_n}}\left(x_{t_1}, x_{t_2}, \ldots, x_{t_n}\right) = F_{X_{t_1+t}, X_{t_2+t}, \ldots, X_{t_n+t}}\left(x_{t_1}, x_{t_2}, \ldots, x_{t_n}\right)$

  $\forall x_i$ $i \in \{1, 2, \ldots, n\}$ and

  $\forall \tau$ such that $(t_i + \tau) \in T$ for all $i$

  - the $k^{th}$ moment of any stationary random process is a constant function of time
    - $EX_{t+t}^k = EX_t^k$ because $F_{X_t}(x) = F_{X_{t+t}}(x)$
    - However, $E\left(X_{t_1} X_{t_2}\right) = R_X\left(t_1, t_2\right) = R_X\left(t_1 - t_2\right) = R_X(t)$

- The random process $\{X_t, t \in T\}$ is **wide sense stationary** (stationary to the second order) if and only if

  1) $E[X_t] = E[X_{t'}] = m_X$ and

  2) $R_X(t, t-\tau) = R_X(\tau)$
     - $= R_X(t+\delta, t+\delta-\tau)\ \forall\delta\forall t = R_X(t', t'-\tau)\ \forall\tau\forall t'$

- Ex. $Z_t = Y\cos(t) + X\sin(t)$

  $EX = EY = 0$

  $X, Y$ are independent $\Rightarrow EXY = EXEY = 0$

  $EX^2 = EY^2 = \sigma^2$

  - $EZ_t = \cos(t)\, EY + \sin(t)\, EX = 0$
  - $R_Z\left(t_1, t_2\right)$

$$= EZ_{t_1}Z_{t_2} = E\left(Y\cos(t_1) + X\sin(t_1)\right)\left(Y\cos(t_2) + X\sin(t_2)\right)$$

$$= \left(EY^2\right)\cos(t_1)\cos(t_2) + \left(EX^2\right)\sin(t_1)\sin(t_2)$$

$$+ \cancel{(EXY)}\left(\cos(t_1)\sin(t_2) + \sin(t_1)\cos(t_2)\right)$$

$$= \left(EY^2\right)\cos(t_1)\cos(t_2) + \left(EX^2\right)\sin(t_1)\sin(t_2)$$

$$= s^2\left(\cos(t_1)\cos(t_2) + \sin(t_1)\sin(t_2)\right) = s^2\cos(t_1 - t_2)$$

- $Z_t$ is stationary in the wide sense, since $EZ_t$ is constant with time, and $R_z(t_1,t_2)$ depends only on the time difference $\tau = t_1 - t_2$.
- Not strictly stationary since $EZ_t^3$ depends on time.

- $R_X(-\tau) = R_X(\tau)$

Proof:

$R_X(\tau) = R_X(t,t-\tau)$

$R_X(-\tau) = R_X(t,t-(-\tau)) = R_X(t,t+\tau) = EX_tX_{t+\tau} = EX_{t+\tau}X_t = R_X(t+\tau,t)$

$= R_X(t+\tau-\tau,t-\tau)$ ; from w.s.s.

$= R_X(t,t-\tau)$

- $\left|R_X(t)\right| \le EX_t^2 = R_X(0)$

Proof :

Cauchy-Schwarz Inequality : $(EXY)^2 \le EX^2EY^2$

$$\left(R_X(t)\right)^2 = \left(EX_tX_{t-t}\right)^2 \le EX_t^2EX_{t-t}^2 \underbrace{=}_{w.s.s.} EX_t^2EX_t^2 = \left(EX_t^2\right)^2$$

$$\left(R_X(t)\right)^2 \le \left(EX_t^2\right)^2 \Rightarrow \left|R_X(t)\right| \le \left|EX_t^2\right| = EX_t^2$$

- If $R_X(\tau)$ of a given process is continuous at $\tau = 0$, then it is also continuous at every other value of $\tau$.

- If $R_X(T) = R_X(0)$, then
  - $X_{t+T} = X_t \ \forall t$

    Proof

    $$E\left(\left|X_{t+T} - X_t\right|^2\right) = EX_{t+T}X_{t+T} + EX_tX_t - 2EX_{t+T}X_t$$

    $$= R_X(0) + R_X(0) - 2R_X(0) = 0$$

    $\sim\!\!\longrightarrow X_t = X_{t+T}$
  - Process is periodic with period T $R_X(\tau+T) = R_X(\tau)$

    Proof: $R_X(t+T) = EX_{t+t+T}X_t = EX_{(t+t)+T}X_t = EX_{t+t}X_t = R_X(t)$

- The random process $\{X_t,\ t \in T\}$ is **covariance stationary** if and only if

  $K_X(t,t-t) = K_X(t',t'-t)\ \forall t = K_X(t)$

- strictly stationary $\rightarrow$ wide sense stationary $\rightarrow$ covariance stationary

- converse does not necessarily hold true

- Proof: strictly stationary → wide sense stationary
  - n = 1: $f_{X_t}(x) = f_{X_{t+t}}(x) \ \forall\tau$

$$EX_{t+t} = \int_{-\infty}^{\infty} x f_{X_{t+t}}(x)\,dx = \int_{-\infty}^{\infty} x f_{X_t}(x)\,dx = EX_t = \mathrm{m_X}$$

  - n = 2: $f_{X_{t_1},X_{t_2}}(x_1,x_2) = f_{X_{t_1+d},X_{t_2+d}}(x_1,x_2) \quad ;\forall d,\forall t_1,\forall t_2$

$$R_X(t_1+d,t_2+d) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1x_2 f_{X_{t_1+d},X_{t_2+d}}(x_1,x_2)\,dx_1dx_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1x_2 f_{X_{t_1},X_{t_2}}(x_1,x_2)\,dx_1dx_2 = R_X(t_1,t_2)\,;\forall d$$

$$\therefore R_X(t_1,t_2) = R_X(t_1-t_2) = R_X(t)$$

- Proof: wide sense stationary → covariance stationary

$$K_X(t_1,t_2) = R_X(t_1,t_2) - \left(EX_{t_1}\right)\left(EX_{t_2}\right) = R_X(t_1-t_2) - m_X^2$$

$$\therefore K_X(t_1,t_2) = K_X(t_1-t_2) = K_X(t)$$

- **<u>If *{X(t)}* is strictly stationary, then *{Y(t)} = {g(X(t)}* is strictly stationary</u>**

- proof:

  Consider finite-dimensional $F_{\underline{Y}(t)}(\underline{y})$

$$F_{\underline{Y}(t)}(\underline{y}) = P\left(\underline{Y}(\underline{t}) \le \underline{y}\right) = P\left(g\left(\underline{X}(\underline{t})\right) \le \underline{y}\right) = P\left(\underline{X}(\underline{t}) \in \left\{\underline{X} : g(\underline{X}) \le \underline{y}\right\}\right)$$

  where $\underline{Y}$, $\underline{t}$, $\underline{y}$, $\underline{X}$, $\underline{x}$ are all n-dimensional vectors.

  Since {X(t)} is strictly stationary,

  for any $\underline{d} = \underbrace{(d,d,\ldots,d)}_{n-\text{dimension}}$,

$$P\left(\underline{X}(\underline{t}) \in \left\{\underline{X} : g(\underline{X}) \le \underline{y}\right\}\right) = P\left(\underline{X}(\underline{t}+\underline{d}) \in \left\{\underline{X} : g(\underline{X}) \le \underline{y}\right\}\right)$$

  Therefore,

$$F_{\underline{Y}(\underline{t})}(\underline{y}) = P\left(\underline{X}(\underline{t}+\underline{d}) \in \left\{\underline{X} : g(\underline{X}) \le \underline{y}\right\}\right)$$

$$= P\left(g\left(\underline{X}(\underline{t}+\underline{d})\right) \le \underline{y}\right) = P\left(\underline{Y}(\underline{t}+\underline{d}) \le \underline{y}\right)$$

$$= F_{\underline{Y}(\underline{t}+\underline{d})}(\underline{y}) \text{ for any } \underline{y},\ \underline{t},\ \underline{\delta}$$

- proof for wide sense stationary

$$E\left[Y(t)\right] = E\left[g\left(X(t)\right)\right] = \int_{-\infty}^{\infty} g(x)\, f_{X(t)}(x)\,dx \text{ and}$$

$$R_Y(t_1,t_2) = E\left[g(X(t_1))\,g(X(t_2))\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x_1)\,g(x_2)\,f_{X(t_1),X(t_2)}(x_1,x_2)\,dx_1dx_2$$

Since {X(t)} is strictly stationary,

$$f_{X(t)}(x) = f_{X(t+d)}(x)$$

$$f_{X(t_1),X(t_2)}(x_1,x_2) = f_{X(t_1+d),X(t_2+d)}(x_1,x_2)$$

for any t, $t_1$, $t_2$, δ

So,

$$E\left[Y(t)\right] = E\left[Y(t+d)\right] \text{ for any t, } \delta$$

$$\Rightarrow E\left[Y(t)\right] = \text{constant for all t}$$

$$R_Y(t_1,t_2) = R_Y(t_1+d, t_2+d) \text{ for any } t_1, t_2, \delta$$

$$\Rightarrow R_Y(t, t-t) = R_Y(t) \text{ for any t and } \tau$$

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Example

$\{X_k, k = 1, 2, \ldots\}$ is collection of i.i.d. $\mathcal{N}(0,\sigma^2)$ r.v.

$\{T_i, i = 1, 2, \ldots\}$ is a collection of i.i.d. exponential r.v. whose common p.d.f. is $f_T(t) = e^{-t}, t \ge 0$

$\{T_i, i = 1, 2, \ldots\}$ is independent of $\{X_k\}$

Let $S_0 = 0$ and $S_k = T_1 + \ldots + T_k$ for $k \ge 1$.

The time-continuous r.p. $\{Z_t, t \ge 0\}$ is defined by $Z_t = X_k$ for $S_{k-1} \le t < S_k$.

That is, $Z_t$ stays at the value $X_1$ over the time interval $[0, T_1)$,
then jumps to the value $X_2$ at which it stays for the next $T_2$ seconds,
then jumps to $X_3$ and stays there for the next $T_3$ seconds, and so on.

- $m_Z(t) = 0$

$$EZ_t = \sum_{k=1}^{\infty} P(S_{k-1} \le t < S_k) EX_k = \sum_{k=1}^{\infty} P(S_{k-1} \le t < S_k) EX$$

$$= EX \sum_{k=1}^{\infty} P(S_{k-1} \le t < S_k) = EX = 0$$

- $R_Z(t,s) = s^2 e^{-|s-t|}$
  - Exponential distribution has no memory: $P(T_k > a+b \mid T_k > a) = P(T_k > b)$

    for a > 0, b > 0

$$P(T_k > a+b \mid T_k > a) = \frac{P(T_k > a+b, T_k > a)}{P(T_k > a)} = \frac{P(T_k > a+b)}{P(T_k > a)} = \frac{e^{-(a+b)}}{e^{-a}} = e^{-b}$$

$$P(T_k > b) = e^{-b}$$

  - Consider the probability that t and s fall into same interval = $p_0$

- When t ≤ s, t must falls into some interval, says $S_{k-1} \le t < S_k$

  $$p_0 = P\left(S_{k-1} \le s < S_k \mid S_{k-1} \le t < S_k\right)$$
  $$= P\left(S_k - S_{k-1} > s - S_{k-1}, s > S_{k-1} \mid S_k - S_{k-1} > t - S_{k-1}, t > S_{k-1}\right)$$

  s ≥ t, so $t > S_{k-1} \rightarrow s > S_{k-1}$

  $$p_0 = P\left(T_k > s - S_{k-1} \mid T_k > t - S_{k-1}\right) = P\left(T_k > s - t + t - S_{k-1} \mid T_k > t - S_{k-1}\right)$$
  $$= P\left(T_k > \underbrace{(s-t)}_{b} + \underbrace{(t - S_{k-1})}_{a} \mid T_k > \underbrace{t - S_{k-1}}_{a}\right) = P\left(T_k > s - t\right) = e^{-(s-t)}$$

- When t > s, $p_0 = P\left(T_k > t - s\right) = e^{-(t-s)}$.
- $p_0 = e^{-|s-t|}$

- When t and s fall into same interval, $Z_t = Z_s$ says $= X_k$.

  This occurs with probability $p_0$.

  When t and s are in different interval, $Z_t \ne Z_s$ says $Z_t = X_{k'}$, $Z_s = X_{k''}$.

  This occurs with probability $1-p_0$.

  $$R_Z(t,s) = EZ_tZ_s = p_0 E\left(X_k X_k\right) + \left(1 - p_0\right) EX_{k'} X_{k''}$$
  $$= p_0 EX_k^{\ 2} + \left(1 - p_0\right) \underbrace{\cancel{EX_{k'}}\ \cancel{EX_{k''}}}_{\text{from independence}} = p_0\left(s_X^{\ 2} + \cancel{m_X^2}\right) = p_0 s_X^{\ 2} = s^{\ 2} e^{-|s-t|}$$

- $\{Z_t\}$ is strictly stationary in the sense that $\{z_t\}$ and $\{W_t\}$ share identical joint distributions, where $W_t = Z_{t+\Delta}$ for any nonnegative time shift Δ.

  Consider n-dimensional joint c.d.f. of $\left\{Z_{t_i}, t = 1, 2, \ldots, n\right\}$

  $$F_{Z_{t_1}, Z_{t_2}, \ldots, Z_{t_n}}\left(z_1, z_2, \ldots, z_n\right)$$

  We can always decompose it as summation of multiplication of one-dimentional p.d.f

  For example, when n =2

  $$F_{Z_{t_1}, Z_{t_2}}\left(z_1, z_2\right) = p_0 F_{X_k}\left(\min\left(z_1, z_2\right)\right) + \left(1 - p_0\right) F_{X_j}\left(z_1\right) F_{X_k}\left(z_2\right) \text{ ; } j \ne k$$

  where $p_0$ = probability that no jump within $(t_1,t_2)$ or $(t_2,t_1)$ = $e^{-|t_1 - t_2|}$.

  Since $\{X_t\}$ is i.i.d., all one dimensional p.d.f. are the same.

  So, joint p.d.f. of $\left\{Z_{t_i}, t = 1, 2, \ldots, n\right\}$ is dependent on $f_X(x)$ and the relation

  between $t_a$ , $t_b$

  We knows that $t_a$ , $t_b$ fall into the same interval or not only depends on the difference between $t_a$ and $t_b$ (Because $T_k$ is exponential distributed.)

  Thus, when we shift $t_i$ to $t_{i+\delta}$ $\forall t_i$ , i = 1,2,…,n, the difference between and two $t_a$, $t_b$ stays the same, i.e. the probability that $t_a$, $t_b$ are in the same interval is unchanged.

  Therefore, $F_{Z_{t_1}, Z_{t_2}, \ldots, Z_{t_n}}\left(z_1, z_2, \ldots, z_n\right) = F_{Z_{t_1+d}, Z_{t_2+d}, \ldots, Z_{t_n+d}}\left(z_1, z_2, \ldots, z_n\right)$.

- $Z_t$ is a gaussian random variable.

$$f_{Z_t}(z) = \sum_{k=1}^{\infty}\left(P\left(S_{k-1} \le t < S_k\right) f_{X_k}(z)\right) = \sum_{k=1}^{\infty}\left(P\left(S_{k-1} \le t < S_k\right) f_X(z)\right)$$

$$= f_X(z)\sum_{k=1}^{\infty} P\left(S_{k-1} \le t < S_k\right) = f_X(z)$$

- $\{Z_t\}$ is not a gaussian random process.

  If $Z_{t_1}, Z_{t_2}$ are jointly gaussian,

$$\Lambda = \begin{bmatrix} s^2 & \operatorname{cov}\left(Z_{t_1}, Z_{t_2}\right) \\ \operatorname{cov}\left(Z_{t_1}, Z_{t_2}\right) & s^2 \end{bmatrix} = \begin{bmatrix} s^2 & s^2 e^{-|t_1 - t_2|} \\ s^2 e^{-|t_1 - t_2|} & s^2 \end{bmatrix} = s^2 \begin{bmatrix} 1 & e^{-|t_1 - t_2|} \\ e^{-|t_1 - t_2|} & 1 \end{bmatrix}$$

$$\operatorname{cov}\left(Z_{t_1}, Z_{t_2}\right) = E\left(Z_{t_1} Z_{t_2}\right) - \cancel{EZ_{t_1}}\,\cancel{EZ_{t_2}} = E\left(Z_{t_1} Z_{t_2}\right) = R_Z\left(t_1, t_2\right) = s^2 e^{-|t_1 - t_2|}$$

$$\Phi_{Z_{t_1}, Z_{t_2}}\left(v_1, v_2\right) = e^{-\underline{v}^T \Lambda \underline{v}} = e^{-\frac{s^2}{2}\left(v_1^2 + v_2^2 + 2 v_1 v_2 e^{-|t_1 - t_2|}\right)}$$

However

$$\Phi_{Z_{t_1}, Z_{t_2}}\left(v_1, v_2\right) = E e^{j\left(v_1 Z_{t_1} + v_2 Z_{t_2}\right)} = p_0 E e^{j\left(v_1 X_k + v_2 X_k\right)} + \left(1 - p_0\right) E e^{j\left(v_1 X_{k'} + v_2 X_{k''}\right)}$$

$$= p_0 E e^{j\left(v_1 + v_2\right) X_k} + \left(1 - p_0\right) E e^{j v_1 X_{k'}} E e^{j v_2 X_{k''}}$$

$$= p_0 \Phi_X\left(v_1 + v_2\right) + \left(1 - p_0\right) \Phi_X\left(v_1\right) \Phi_X\left(v_2\right)$$

$$= p_0 e^{-\frac{\left(v_1 + v_2\right)^2 s^2}{2}} + \left(1 - p_0\right) e^{-\frac{v_1^2 s^2}{2}} e^{-\frac{v_2^2 s^2}{2}}$$

$$= e^{-|t_1 - t_2|} e^{-\frac{\left(v_1 + v_2\right)^2 s^2}{2}} + \left(1 - e^{-|t_1 - t_2|}\right) e^{-\frac{v_1^2 s^2}{2}} e^{-\frac{v_2^2 s^2}{2}}$$

$\Phi_{Z_{t_1}, Z_{t_2}}\left(v_1, v_2\right)$ does not have the form of jointly gaussian characteristic function. So, $\{Z_t\}$ is not a gaussian random process.

## Gaussian Random Variable

- gaussian random variable X with mean m and variance $\sigma^2$ :

$$f_X(x) = \frac{1}{\sqrt{2p}\, s} e^{-\frac{(x-m)^2}{2s^2}}$$

- random variables $X_1$, $X_2$, …, $X_n$ are jointly gaussian random variables iff

$$f_{X_{t_1}, X_{t_2}, \ldots X_{t_n}}\left(x_1, x_2, \ldots, x_n\right) = \frac{1}{(2p)^{\frac{n}{2}} |\Lambda|^{\frac{1}{2}}} e^{-\frac{1}{2|\Lambda|}\sum_{i=1}^{n}\sum_{j=1}^{n} |\Lambda|_{ij}\left(x_i - m_i\right)\left(x_j - m_j\right)}$$

where

$m_i = EX_{t_i}$ ; i = 1, 2, …, n

$\Lambda$ = covariance matrix = $\left(\lambda_{ij}\right)_{n\times n}$ = $\left(\operatorname{cov}\left(X_{t_i}, X_{t_j}\right)\right)_{n\times n}$ = $\left(K_X\left(t_i, t_j\right)\right)_{n\times n}$

$\left|\Lambda\right|_{ij}$ = cofactor of the element $\lambda_{ij}$ in the determinant $|\Lambda|$

Assumed that $\Lambda$ is nonsingular

- $X_1$ and $X_2$ are jointly Gaussian random variables, if and only if

$$f_{X_1,X_2}\left(x_1,x_2\right)=\frac{1}{2\pi\sigma_{X_1}\sigma_{X_2}\sqrt{1-\rho^2}}e^{-\frac{\left(\frac{x_1-EX_1}{\sigma_{X_1}}\right)^2-2\rho\left(\frac{x_1-EX_1}{\sigma_{X_1}}\right)\left(\frac{x_2-EX_2}{\sigma_{X_2}}\right)+\left(\frac{x_2-EX_2}{\sigma_{X_2}}\right)^2}{2\left(1-\rho^2\right)}},$$

where $\rho = \dfrac{Cov\left(X_1,X_2\right)}{\sigma_{X_1}\sigma_{X_2}} = \dfrac{E\left(X_1X_2\right)-EX_1EX_2}{\sigma_{X_1}\sigma_{X_2}}$

- If $EX_1 = EX_2 = 0$ and $\sigma_{X_1} = \sigma_{X_2} = \sigma$

$$f_{X_1,X_2}\left(x_1,x_2\right)=\frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}}e^{-\frac{x_1^2-2\rho x_1x_2+x_2^2}{2\sigma^2\left(1-\rho^2\right)}}$$

$$\rho = \frac{E\left(X_1X_2\right)}{\sigma^2} = \frac{R_X\left(t\right)}{\sigma^2}$$

- If $Cov\left(X_1,X_2\right) = 0$ (or $X_1 \perp\!\!\!\perp X_2$ which also make $Cov\left(X_1,X_2\right) = 0$)

$$f_{X_1,X_2}\left(x_1,x_2\right)=f_{X_1}\left(x_1\right)\times f_{X_2}\left(x_2\right)=\frac{1}{2\pi\sigma_{X_1}\sigma_{X_2}}e^{-\frac{\left(\frac{x_1-EX_1}{\sigma_{X_1}}\right)^2+\left(\frac{x_2-EX_2}{\sigma_{X_2}}\right)^2}{2}}$$

$Cov\left(X_1,X_2\right) = 0 \rightarrow \rho = 0$

$$f_{X_1,X_2}\left(x_1,x_2\right)=\frac{1}{2\pi\sigma_{X_1}\sigma_{X_2}\sqrt{1-\rho^2}}e^{-\frac{\left(\frac{x_1-EX_1}{\sigma_{X_1}}\right)^2-2\rho\left(\frac{x_1-EX_1}{\sigma_{X_1}}\right)\left(\frac{x_2-EX_2}{\sigma_{X_2}}\right)+\left(\frac{x_2-EX_2}{\sigma_{X_2}}\right)^2}{2\left(1-\rho^2\right)}}$$

$$=\frac{1}{2\pi\sigma_{X_1}\sigma_{X_2}}e^{-\frac{\left(\frac{x_1-EX_1}{\sigma_{X_1}}\right)^2+\left(\frac{x_2-EX_2}{\sigma_{X_2}}\right)^2}{2}}$$

$$= f_{X_1}\left(x_1\right)\times f_{X_2}\left(x_2\right)$$

## Gaussian Random Process

- A real random process $\{X_t\}$ is a **gaussian random process** if and only if for every n (given any arbitrary set of n time instants $t_1, t_2, \ldots, t_n$ in the index set T), the random variable $X_{t_1}, X_{t_2}, \dots, X_{t_n}$

- are jointly gaussian random variables
  - form the components of an n-dimensional gaussian random vector

- **If a real gaussian random process is w.s.s., then it is also s.s.**

Proof:

From w.s.s.,

$$EX_{t_i} = m_j = m \text{ constant}$$

$$K_X(t_i, t_j) = R_X(t_i, t_j) - EX_{t_i} EX_{t_j} = R_X(t_i - t_j) - m^2$$

$$f_{X_{t_1}, X_{t_2}, \dots X_{t_n}}(x_1, x_2, \dots, x_n) = \frac{1}{(2p)^{\frac{n}{2}} |\Lambda|^{\frac{1}{2}}} e^{-\frac{1}{2|\Lambda|}\sum_{i=1}^{n}\sum_{j=1}^{n} |\Lambda|_{ij}(x_i - m_i)(x_j - m_j)}$$

Then, $f_{X_{t_1}, X_{t_2}, \dots, X_{t_n}}(x_{t_1}, x_{t_2}, \dots, x_{t_n}) = f_{X_{t_1+t}, X_{t_2+t}, \dots, X_{t_n+t}}(x_{t_1}, x_{t_2}, \dots, x_{t_n})$

- **Bussgang's Theorem**

Input: {X(t)} is a **zero-mean stationary Gaussian random process**.

Output: {Y(t)} is defined by Y(t) = g(X(t))

g(·) is an instantaneous nonlinear device (i.e., an ordinary real-valued function of a real variable).

→

- {Y(t)} is not necessarily Gaussian.
- {Y(t)} is a strictly stationary random process
- $R_{XY}(\tau) = C\, R_X(\tau)$, where C = $\frac{1}{s^2} E\left(X_t \cdot g(X_t)\right)$ constant not depends on τ

⇒ If the input to an instantaneous nonlinear device is a stationary Gaussian random process, then the crosscorrelation function of the input and output is proportional to the auto correlation function of the input

Proof:

$X_1$ and $X_2$ are jointly Gaussian random variables, if and only if

$$f_{X_{t_1}, X_{t_2}}(x_1, x_2) = \frac{1}{2p s_{X_1} s_{X_2} \sqrt{1 - r^2}} e^{-\frac{\left(\frac{x_1 - EX_1}{s_{X_1}}\right)^2 - 2r\left(\frac{x_1 - EX_1}{s_{X_1}}\right)\left(\frac{x_2 - EX_2}{s_{X_2}}\right) + \left(\frac{x_2 - EX_2}{s_{X_2}}\right)^2}{2(1 - r^2)}},$$

where $r = \frac{Cov(X_1, X_2)}{s_{X_1} s_{X_2}} = \frac{E(X_1 X_2) - EX_1 EX_2}{s_{X_1} s_{X_2}}$

In this problem,

$EX_1 = EX_2 = 0$

From stationaryness, $s_{X_1} = s_{X_2} = s$

$$f_{X_1,X_2}(x_1,x_2)=\frac{1}{2ps^2\sqrt{1-r^2}}e^{-\frac{x_1^2-2rx_1x_2+x_2^2}{2s^2(1-r^2)}}$$

$$r=\frac{E(X_1X_2)}{s^2}=\frac{R_X(t)}{s^2}$$

$$R_{X,Y}(t_1,t_2)$$

$$=E\left(X_{t_1},Y_{t_2}\right)=E\left(X_{t_1},g\left(X_{t_2}\right)\right)$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x_1g(x_2)f_{X_{t_1},X_{t_2}}(x_1,x_2)\,dx_1dx_2$$

$$=\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}x_1g(x_2)\frac{1}{2ps^2\sqrt{1-r^2}}e^{-\frac{x_1^2-2rx_1x_2+x_2^2}{2s^2(1-r^2)}}dx_1\right)dx_2$$

$$=\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}x_1g(x_2)\frac{1}{2ps^2\sqrt{1-r^2}}e^{-\frac{(x_1-rx_2)^2+(1-r^2)x_2^2}{2s^2(1-r^2)}}dx_1\right)dx_2$$

$$=\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}x_1g(x_2)\frac{1}{2ps^2\sqrt{1-r^2}}e^{-\frac{(x_1-rx_2)^2}{2s^2(1-r^2)}}e^{-\frac{x_2^2}{2s^2}}dx_1\right)dx_2$$

$$=\int_{-\infty}^{\infty}g(x_2)e^{-\frac{x_2^2}{2s^2}}\left(\frac{\sqrt{2p}\sqrt{s^2(1-r^2)}}{2ps^2\sqrt{1-r^2}}\underbrace{\int_{-\infty}^{\infty}x_1\underbrace{\frac{1}{\sqrt{2p}\sqrt{s^2(1-r^2)}}e^{-\frac{(x_1-rx_2)^2}{2(s^2(1-r^2))}}}_{\text{Gaussian } m'=rx_2,\ s'^2=s^2(1-r^2)}dx_1}_{m'}\right)dx_2$$

$$=\int_{-\infty}^{\infty}g(x_2)e^{-\frac{x_2^2}{2s^2}}\left(\frac{1}{\sqrt{2p}\,s}rx_2\right)dx_2=r\int_{-\infty}^{\infty}x_2g(x_2)\frac{1}{\sqrt{2p}\,s}e^{-\frac{x_2^2}{2s^2}}dx_2$$

$$=R_X(t)\frac{1}{s^2}\int_{-\infty}^{\infty}x_2g(x_2)\frac{1}{\sqrt{2p}\,s}e^{-\frac{x_2^2}{2s^2}}dx_2$$

Therefore, C = $\frac{1}{s^2}\int_{-\infty}^{\infty}x_2g(x_2)\frac{1}{\sqrt{2p}\,s}e^{-\frac{x_2^2}{2s^2}}dx_2 = \frac{1}{s^2}E\left(X_t\cdot g\left(X_t\right)\right)$

- A w.s.s. (strictly stationary) continuous-time Gaussian r.p. X(t) is Markovian if and only if $K_X(t)=K_X(0)e^{-a|t|}$,

  where α is real and positive

- For 0-mean, can use $R_X(t) = R_X(0)e^{-a|t|}$

# Linear Transformation

## Linear transformations on finite-dimensional random vectors

- $\underline{Y} = \underline{g}(\underline{X})$ = $g_{m\times n}\underline{X}$ where dim(X) = n ≥ m = dim(Y)

  transformation g is a linear transformation iff $\underline{g}(a\underline{W} + b\underline{Z}) = a\underline{g}(\underline{W}) + b\underline{g}(\underline{Z})$

  where

  a and b are arbitrary real constants and

  $\underline{W}$ and $\underline{Z}$ are arbitrary real random vectors

- General linear transformation : $Y_i = \sum_{j=1}^{n} g_{ij} X_j$ where i = 1, 2, …, m
  - $EY_i = \sum_{j=1}^{n} g_{ij} EX_j$
  - $\operatorname{cov}(Y_i, Y_k) = \sum_{j=1}^{n}\sum_{r=1}^{n} g_{ij} g_{kr} \operatorname{cov}(X_j, X_r)$
  - $s_{Y_i}^2 = \operatorname{cov}(Y_i, Y_i) = \sum_{j=1}^{n}\sum_{r=1}^{n} g_{ij} g_{ir} \operatorname{cov}(X_j, X_r)$

- $E\underline{Y} = gE\underline{X}$
- $\Lambda_{\underline{Y}} = g\Lambda_{\underline{X}} g^T$
  - $\Lambda_{\underline{X}} = \left(\operatorname{cov}(X_i, X_j)\right)$
  - $\Lambda_{\underline{Y}} = \left(\operatorname{cov}(Y_i, Y_j)\right)$

Example: Linear transformation for two-dimensional real random vector $\underline{X}$

$Y_1 = g_{11}X_1 + g_{12}X_2$

$Y_2 = g_{21}X_1 + g_{22}X_2$

- If g has an inverse $\left(\begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix} \neq 0\right)$, say $\underline{h} = (h_1, h_2)$, then

$$f_{Y_1,Y_2}(y_1, y_2) = \left|\frac{\partial(h_1, h_2)}{\partial(y_1, y_2)}\right| f_{X_1,X_2}\left(h_1(y_1, y_2), h_2(y_1, y_2)\right)$$

- $EY_1 = g_{11}EX_1 + g_{12}EX_2$

  $EY_2 = g_{21}EX_1 + g_{22}EX_2$

- $s_{Y_1}^{2} = g_{11}^{2} s_{X_1}^{2} + g_{12}^{2} s_{X_2}^{2} + 2 g_{11} g_{12} \operatorname{cov}(X_1, X_2)$

  $s_{Y_2}^{2} = g_{21}^{2} s_{X_1}^{2} + g_{22}^{2} s_{X_2}^{2} + 2 g_{21} g_{22} \operatorname{cov}(X_1, X_2)$

$$
\begin{aligned}
s_{Y_1}^{2} &= E\left(Y_1 - EY_1\right)^2 = E\left(\left(g_{11}X_1 + g_{12}X_2\right) - \left(g_{11}EX_1 + g_{12}EX_2\right)\right)^2 \\
&= E\left(g_{11}\left(X_1 - EX_1\right) + g_{12}\left(X_2 - EX_2\right)\right)^2 \\
&= g_{11}^2 E\left(X_1 - EX_1\right)^2 + g_{12}^2 E\left(X_2 - EX_2\right)^2 \\
&\quad + 2 g_{11} g_{12} E\left(X_1 - EX_1\right)\left(X_2 - EX_2\right) \\
&= g_{11}^2 s_{X_1}^{2} + g_{12}^2 s_{X_2}^{2} + 2 g_{11} g_{12} \operatorname{cov}(X_1, X_2)
\end{aligned}
$$

- $\operatorname{cov}(Y_1, Y_2) = g_{11} g_{21}\left(s_{X_1}^{2}\right) + g_{12} g_{22}\left(s_{X_2}^{2}\right) + \left(g_{11} g_{22} + g_{12} g_{21}\right) \operatorname{cov}(X_1, X_2)$

$$
\begin{aligned}
& EY_1Y_2 = E\left(g_{11}X_1 + g_{12}X_2\right)\left(g_{21}X_1 + g_{22}X_2\right) \\
& = g_{11}g_{21}EX_1^2 + g_{12}g_{22}EX_2^2 + \left(g_{11}g_{22} + g_{12}g_{21}\right)EX_1X_2 \\
& = g_{11}g_{21}\left(s_{X_1}^{2} + \left(EX_1\right)^2\right) + g_{12}g_{22}\left(s_{X_2}^{2} + \left(EX_2\right)^2\right) \\
& \quad + \left(g_{11}g_{22} + g_{12}g_{21}\right)\left(\operatorname{cov}(X_1, X_2) + EX_1EX_2\right) \\
& EY_1EY_2 = \left(g_{11}EX_1 + g_{12}EX_2\right)\left(g_{21}EX_1 + g_{22}EX_2\right) \\
& = g_{11}g_{21}\left(EX_1\right)^2 + g_{12}g_{22}\left(s_{X_2}^{2} + \left(EX_2\right)^2\right) + \left(g_{11}g_{22} + g_{12}g_{21}\right)\left(EX_1EX_2\right) \\
& \operatorname{cov}(Y_1, Y_2) = EY_1Y_2 - EY_1EY_2 \\
& = g_{11}g_{21}\left(s_{X_1}^{2}\right) + g_{12}g_{22}\left(s_{X_2}^{2}\right) + \left(g_{11}g_{22} + g_{12}g_{21}\right)\operatorname{cov}(X_1, X_2)
\end{aligned}
$$