Renewal Theory

- The counting process $\{A(t)\}$, where t may be either discrete or continuous, is a renewal process if,

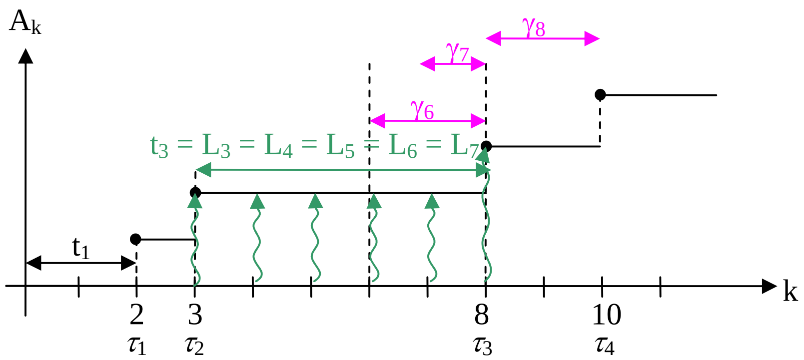
  given the count has increased at time t^*,

  the process of <u>future</u> increments $\{A(s)-A(t^*), s>t^*\}$ and the process of <u>past</u> increments $\{A(t^*)-A(s), s<t^*\}$ are statistically independent.

- Markov counting processes, including both homogeneous and inhomogeneous Poison processes, are special cases of renewal processes.

  For any Markov process, be it a counting process or not, we have independence of $\{A(s), s>t | A(t)\}$ from $\{A(s), s<t | A(t)\}$ for **<u>every</u>** t, and hence a fortiori independence of $\{A(s)-A(t^*), s>t^*\}$ and $\{A(t^*)-A(s), s<t^*\}$

Discrete Time Renewal Processes



- $A_k = \max\{n : \tau_n \le k\}$ = number of renewals up to and including time k

  At jump, A_{k^*} takes the jumped (higher) value

  t_i = the i^{th} gap length between renewals, $i \ge 2$

  $\tau_n = t_1 + t_2 + \ldots + t_n$ = time of the n^{th} renewal

  $\gamma_k = \tau_{A_k+1} - k$ = **residual lifetime** at time k

  $L_k = \tau_{A_k+1} - \tau_{A_k} = t_{A_k+1}$ = **selected lifetime** at time k

  - At jump, $\gamma_{k^*} = L_{k^*} = t_{A_{k^*}+1}$
  - Note that $\gamma_k, L_k \in \mathbb{N}$. Also, $\gamma_k \le L_k$.

- Statistical assumptions
  - $t_1, t_2, \ldots$ are independent
  - t_1 has probability vector $\underline{g} = (g_0, g_1, \ldots)$

    $g_j = P(t_1 = j)$

- t_i for $i \ge 2$ has probability vector $\underline{f} = (f_1, f_2, \ldots)$ i.i.d.

  $f_j = P(t_i = j)$; $\forall i \ge 2$

  moments of the distribution f

  $m_n = Et_i^n$ for $i \ge 2$, $n = 1, 2, 3, \ldots$

  We have assumed $f_0 = 0$ in order to preclude the possibility of batch arrivals in our renewal counting process.

- **Renewal distribution** $\Rightarrow h_k, k \ge 0 \Rightarrow h_k = P(\text{a renewal occurs at time k})$
  - **<u>Not</u>** a probability distribution. Not sum to 1 in general. In most case of interest, sum $\to \infty$.

- **Renewal equation**: $h_k = g_k + \sum_{j=0}^{k-1} h_j f_{k-j}$; $k \ge 0$
  - $h_0 = g_0 + 0$

    For $k = 0$, the sum is empty
  - $h_k = P(\text{a renewal occurs at time k})$ is the sum of 1) the probability that this renewal at k is the first renewal and 2) the sum for all j probability that the last renewal occur at j, and then this one at k is the next renewal without any renewal occurs in between.
  - The right-hand side of the renewal equation breaks down all the instances in which there is a renewal at time k into disjoint sets, the j^{th} of which contains all those cases in which the renewal that immediately precedes the one at time k takes place at time $j < k$.
  - Alternative form: $h_k = g_k + \sum_{i=1}^{k} f_i h_{k-i}$; $k \ge 0$

    derived by substituting $i = k\text{-}j$

- $\{A(k)\}$ is not a Markov chain unless the t_i are geometrically distributed.

  This is because, in other than geometrically distributed cases, the residual lifetime γ_k at time k is statistically dependent on the time $L_k\text{-}\gamma_k$ that has elapsed since the most recent renewal.

- Theorem:

  1) $\{(\gamma_k, L_k), k \ge 0\}$ is a homogeneous Markov chain.
  2) If $GCD\{k : f_k \ne 0\} = 1$, then this Markov chain is aperiodic.
  3) If, in addition, $m_1 < \infty$, the chain is ergodic.

     Its equilibrium distribution, which is a limiting distribution in the ergodic case, is

$$\boxed{P\left[(\gamma, L) = (i, j)\right] = P(i, j) = \begin{cases} \dfrac{f_j}{m_1} & \text{if } 1 \le i \le j \\ 0 & \text{otherwise} \end{cases}}$$

- Remark
  - $P(L = j) = \dfrac{jf_j}{m_1}$

    Proof. $P(L = j) = \sum_{i=1}^{j} P\left[(\gamma, L) = (i, j)\right] = \sum_{i=1}^{j} \dfrac{f_j}{m_1} = j\dfrac{f_j}{m_1}$
  - Given that $L = j$, γ is uniformly distributed over $\{1, 2, \ldots, j\}$

    Proof. $P(\gamma = i | L = j) = \dfrac{P\left[(\gamma, L) = (i, j)\right]}{P(L = j)} = \begin{cases} \dfrac{f_j}{m_1} \Big/ j\dfrac{f_j}{m_1} & \text{if } 1 \le i \le j \\ 0 \Big/ j\dfrac{f_j}{m_1} & \text{otherwise} \end{cases}$

    $$= \begin{cases} \dfrac{1}{j} & \text{if } 1 \le i \le j \\ 0 & \text{otherwise} \end{cases}$$

  - Hence, it may help to think of $P\left[(\gamma, L) = (i, j)\right] = \dfrac{f_j}{m_1}$ as

    $P(\gamma = i, L = j) = P(\gamma = i | L = j)\underbrace{P(L = j)}_{\text{choose gap}} = \dfrac{1}{j}\dfrac{jf_j}{m_1} = \dfrac{f_j}{m_1}$.
  - $\boxed{P(\gamma = i) = \dfrac{1}{m_1}\sum_{j=i}^{\infty} f_j = \dfrac{1 - F(i-1)}{m_1}}$; $F(m) = \sum_{n=1}^{m} f_n$

    Proof. $P(\gamma = i) = \sum_{j=1}^{\infty} P(i, j) = \sum_{j=1}^{\infty} \dfrac{f_j}{m_1} = \sum_{j=i}^{\infty} \dfrac{f_j}{m_1}$

    $$= \frac{f_i + f_{i+1} + \ldots}{m_1} = \frac{1 - (f_1 + f_2 + \ldots f_{i-1})}{m_1} = \frac{1 - F(i-1)}{m_1}$$
  - $E[\gamma] = \dfrac{m_2 + m_1}{2m_1}$

    Proof. $E\left[\gamma^k\right] = \sum_{i=1}^{\infty} i^k P(\gamma = i) = \sum_{i=1}^{\infty} i^k \dfrac{1}{m_1} \sum_{j=i}^{\infty} f_j = \dfrac{1}{m_1} \sum_{i=1}^{\infty} \sum_{j=i}^{\infty} i^k f_j = \dfrac{1}{m_1} \sum_{j=1}^{\infty} f_j \sum_{i=1}^{j} i^k$

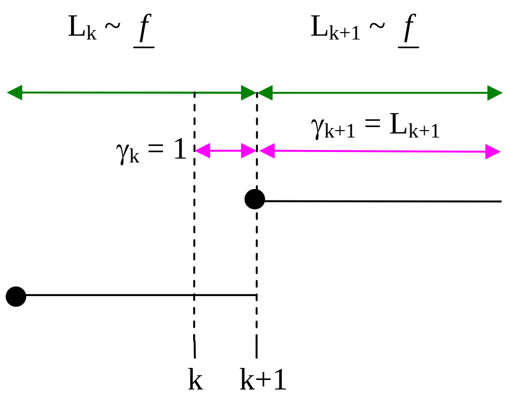
For $k = 1$, $E[\gamma] = \frac{1}{m_1}\sum_{j=1}^{\infty} f_j \sum_{i=1}^{j} i = \frac{1}{2m_1}\sum_{j=1}^{\infty} f_j\left(j^2 + j\right) = \frac{m_2 + m_1}{2m_1}$.

Proof of the theorem:

Consider the case when $\gamma_k = 1$, then L_k will end at time k+1, where upon L_{k+1} will be chosen according to the distribution $\underline{f}$, and $\gamma_{k+1} = L_{k+1}$. Notationally, $(\gamma_k, L_k) = (1, j) \Rightarrow \gamma_{k+1} = L_{k+1} \sim \underline{f}$.

Therefore, $P_{\gamma_{k+1}, L_{k+1}|\gamma_k, L_k}\left(i', j' \middle| i, j\right) = f_{j'}\delta_{i',j'}$ for $i = 1$

$\Rightarrow$ given that $\gamma_k = 1$, then $\gamma_{k+1} = i'$ has to equal $L_{k+1} = j'$, and the probability that they are not equal is 0, thus having $\delta_{i',j'}$.



On the other hand, if $\gamma_k > 1$, then $L_{k+1} = L_k$ and $\gamma_{k+1} = \gamma_k - 1$ (one time step closer). $(\gamma_k, L_k) = (i > 1, j) \Rightarrow (\gamma_{k+1}, L_{k+1}) = (i-1, j)$.

Since we want $j' = j$ and $i' = i - 1$,

$$P_{\gamma_{k+1}, L_{k+1}|\gamma_k, L_k}\left(i', j' \middle| i, j\right) = \delta_{j',j}\delta_{i',i-1} \quad \text{for } i > 1.$$

So, the general entry in one-step transition matrix is

$$P_{\gamma_{k+1}, L_{k+1}|\gamma_k, L_k}\left(i', j' \middle| i, j\right) = \begin{cases} f_{j'}\delta_{i',j'} & \text{for } i = 1 \\ \delta_{j',j}\delta_{i',i-1} & \text{for } i > 1 \end{cases} = f_{j'}\delta_{i',j'}\delta_{i,1} + \delta_{j',j}\delta_{i',i-1}\left(1 - \delta_{i,1}\right).$$

Note that it does not depend on the time k, so the Markov chain is homogeneous. To verify that the distribution $P(i, j)$ in the theorem statement is the equilibrium distribution corresponding to the transition matrix $P\left(i', j' \middle| i, j\right)$, we need to show that

1) $\sum_{(i,j)} P(i, j) = 1$, and

2) $\sum_{(i,j)} P\left(i', j' \middle| i, j\right) P(i, j) = P(i', j')$

Property (1) is trivial because

$$\sum_{(i,j)} P(i,j) = \sum_{j=1}^{\infty} \sum_{i=1}^{j} \frac{f_j}{m_1} = \frac{1}{m_1} \sum_{j=1}^{\infty} j f_j = \frac{m_1}{m_1} = 1.$$

To prove (2), let $g(i',j') = \sum_{(i,j)} P(i',j'|i,j) P(i,j)$.

$$\begin{aligned}
g(i',j') &= \sum_{(i,j)} P(i',j'|i,j) P(i,j) \\
&= \sum_{j=1}^{\infty} \sum_{i=1}^{j} \left(f_{j'} \delta_{i',j'} \delta_{i,1} + \delta_{j',j} \delta_{i',i-1} (1-\delta_{i,1}) \right) \frac{f_j}{m_1} \\
&= \sum_{j=1}^{\infty} \frac{f_j}{m_1} \left(f_{j'} \delta_{i',j'} \sum_{i=1}^{j} \delta_{i,1} + \delta_{j',j} \sum_{i=1}^{j} \delta_{i',i-1} (1-\delta_{i,1}) \right) \\
&= \sum_{j=1}^{\infty} \frac{f_j}{m_1} \left(f_{j'} \delta_{i',j'} + \delta_{j',j} \sum_{i=1}^{j} \delta_{i',i-1} (1-\delta_{i,1}) \right) \\
&= \frac{f_{j'}}{m_1} \delta_{i',j'} + \frac{f_{j'}}{m_1} \sum_{i=1}^{j'} \delta_{i',i-1} (1-\delta_{i,1})
\end{aligned}$$

The first term will be zero if $i' \neq j'$. The second terms will be zero if $i' \geq j'$. To see this, assume $i' \geq j'$. Note that the sum goes from *i* = 1 to *j*′. So,

$$0 \leq i-1 \leq j'-1 < j' \leq i',$$

i.e. $i-1$ will never equal to (always less than) i'. Because of $\delta_{i',i-1}$, the sum is zero. Hence, we conclude that $g(i',j')$ = (*a*) $\frac{f_{j'}}{m_1}$ if $i' = j'$, (*b*) $\frac{f_{j'}}{m_1} \sum_{i=1}^{j'} \delta_{i',i-1} (1-\delta_{i,1})$ if $i' < j'$, and (c) 0 if $i' > j'$.

Note the factor of $f_{j'}$. This implies that the result is zero for $j' < 1$. So we concern only with $j' \geq 1$.

Consider the sum $\frac{f_{j'}}{m_1} \sum_{i=1}^{j'} \delta_{i',i-1} (1-\delta_{i,1})$. Because *i* is between 1 and j', if $i' < 0$ there is no *i* such that $i-1 = i'$. This is because $0 \leq i-1 \leq j'$ and $j' \geq 1$. Hence, the cases when $i' < 0$ yields zero due to the existence of $\delta_{i',i-1}$. The case when $i' = 0$ also yields zero because $\delta_{i',i-1}$ requires *i* = 1 which would make $1-\delta_{i,1}$ zero. So, nonzero result is possible only when $i' \geq 1$. Combining $i' < j'$ and $i' \geq 1$, we know that there exists *i* such that $i-1 = i'$ and *i* ≠ 1; hence,

$$\frac{f_{j'}}{m_1} \sum_{i=1}^{j'} \delta_{i',i-1} (1-\delta_{i,1}) = \frac{f_{j'}}{m_1}.$$

Finally we have

(a) when $i' = j'$, $g(i',j') = \frac{f_{j'}}{m_1}$ if $1 \le i' = j'$, 0 otherwise.

(b) when $i' < j'$, $g(i',j') = \frac{f_{j'}}{m_1}$ if $j', i' \ge 1$, 0 otherwise.

(c) when $i' > j'$, $g(i',j') = 0$.

This is the same as saying $g(i',j') = \begin{cases} \frac{f_{j'}}{m_1} & \text{if } 1 \le i' \le j' \\ 0 & \text{otherwise} \end{cases}$, which equals $P(i',j')$ as was to be shown.

Some facts about Continuous Time Renewal Processes

- Def:

  $t_1, t_2, \ldots$ are independent, t_1 has cdf *G*, and the t_k for $k \ge 2$ are i.i.d. with cdf *F*.

  $\tau_n = \sum_{k=1}^{n} t_k$ denote the time of occurrence of the n^{th} renewal.

  The renewal counting process $\{A(t), t \ge 0\}$ is defined by $A_t = \max\{n : \tau_n \le t\}$.

  The residual lifetime at *t* is $\gamma_t = \tau_{A_t+1} - t$.

  The selected lifetime at *t* is $L_t = \tau_{A_t+1} - \tau_{A_t} = t_{A_t+1}$.

  $H(t) = E[A_t]$ = Expected number of renewals up to (and including) time *t*.

  m_k = the k^{th} moment of *F*.

- Continuous time renewal equation: rate $\frac{d}{dt}H(t) = h(t) = g(t) + \int_0^t h(t-s)\, f(s) ds$.

- A distribution is said to be lattice if its points of increase are contained in the set of integral multiples of some real number *r*; otherwise it is non-lattice.

- Blackwell's Renewal Theorem: Suppose *F* is non-lattice. The for fixed $h > 0$,

  $$\lim_{t\to\infty} \frac{H(t+h) - H(t)}{h} = \frac{1}{m_1}.$$

  Loosely stated, Blackwell's theorem says that in non-lattice cases the renewal process eventually "forgets" about initial conditions (i.e., about where the time origin is) in the sense that, at any randomly chosen time *t* in the remote future, renewals are occurring in the vicinity of *t* at a rate versus time of m_1^{-1}.

- $\lim_{t\to\infty} \Pr[\gamma_t \le x, L_t \le y] = F_{\gamma,L}(x,y)$ exists.

- $f_\gamma(x) = \frac{1 - F(x)}{m_1}$, $x \ge 0$. If *F* has density *f*, then $f_\gamma(x) = \int_x^\infty \frac{f(y)}{m_1} dy$.

- If F has density f, then $f_{\gamma,L}(x,y) = \begin{cases} \dfrac{f(y)}{m_1}, & 0 \le x \le y \\ 0, & \text{otherwise} \end{cases}$.
- If F has density f, then $f_L(y) = \dfrac{yf(y)}{m_1}$, $y \ge 0$

  Proof. $f_L(y) = \int_{-\infty}^{\infty} f_{\gamma,L}(x,y)dx = \begin{cases} \int_0^y \dfrac{f(y)}{m_1}dx, & y \ge 0 \\ 0, & \text{otherwise} \end{cases} = \begin{cases} \dfrac{yf(y)}{m_1}, & y \ge 0 \\ 0, & \text{otherwise} \end{cases}$
- $E\left[L^k\right] = \dfrac{m_{k+1}}{m_1}$

  Proof. $E\left[L^k\right] = \int_{-\infty}^{\infty} y^k f_L(y)dy = \int_0^{\infty} y^k \dfrac{yf(y)}{m_1}dy = \dfrac{m_{k+1}}{m_1}$
- $E\left[\gamma^k\right] = \dfrac{m_{k+1}}{(k+1)m_1}$. More specifically, $\boxed{E[\gamma] = \dfrac{1}{2}\dfrac{E\left[t^2\right]}{E[t]}}$.

  Proof. $E\left[\gamma^k\right] = \int_{-\infty}^{\infty} x^k f_\gamma(x)dx = \int_0^{\infty} x^k \int_x^{\infty} \dfrac{f(y)}{m_1}dydx = \int_0^{\infty} \dfrac{f(y)}{m_1}\int_0^y x^k dxdy$

  $= \dfrac{1}{m_1}\int_0^{\infty} f(y)\dfrac{y^{k+1}}{k+1}dy = \dfrac{m_{k+1}}{(k+1)m_1}$
- We shall assume the $F(0) = 0$, meaning that there is zero probability that no gap occurs between renewals; this is consistent with our having ruled out multiple simultaneous renewals in discrete time.
- Laplace-Stieltjes Transform: If $F_B(t)$ is the cdf of a nonnegative random variable, then its L-S transform $L_g(s)$ is defined by $L_B(s) = \int_0^{\infty} e^{-st}dF_B(t)$.

  In the event that B has a density $f_B(t) = \dfrac{d}{dt}F_B(t)$, then $L_B(s) = \int_0^{\infty} e^{-st} f_B(t)dt$.

  Note that $\dfrac{d}{ds}L_B(s) = \dfrac{d}{ds}\int_0^{\infty} e^{-st}dF_B(t) = \int_0^{\infty} \dfrac{d}{ds}e^{-st}dF_B(t) = -\int_0^{\infty} te^{-st}dF_B(t)$.
- $L_\gamma(s) = \dfrac{1 - L_F(s)}{sm_1}$

Proof. $L_\gamma(s)=\int_0^\infty e^{-st}\frac{1-F(t)}{m_1}dt=-\frac{1}{sm_1}e^{-st}\Big|_0^\infty-\frac{1}{m_1}\left(\cancel{-\frac{1}{s}F(t)e^{-st}\Big|_0^\infty}^{\,0}+\int_0^\infty\frac{1}{s}f(t)e^{-st}dt\right)$

$$=\frac{1}{sm_1}-\frac{1}{m_1}\frac{1}{s}L_F(s)$$

- $L_L(s)=-\frac{1}{m_1}\frac{d}{ds}L_B(s)$

Proof. $L_L(s)=\int_0^\infty e^{-st}dF_L(t)=\frac{1}{m_1}\int_0^\infty e^{-st}tf(t)dt=-\frac{1}{m_1}\frac{d}{ds}L_B(s)$.