Poisson Processes

- Used to model phenomena that occur at "purely random" instants in time.
  - Characterization II supports this interpretation strongly.

- Define $\{X(t), t \in T\}$, where T is an interval of the real line (often $[0,\infty)$ or $(-\infty,\infty)$)

  Def: $\{X(t)\}$ is a **<u>homogeneous Poisson process with rate λ</u>**

  if it is a homogeneous Markov process with

  state space $S = \{0, \pm1, \pm2, \ldots\}$ (or sometimes $S = \{0, 1, 2, \ldots\}$) and

  transition rate matrix $q_{i,j} = \begin{cases} \lambda & \text{if } j = i+1 \\ -\lambda & \text{if } j = i \\ 0 & \text{otherwise} \end{cases}$

- $$Q = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & \cdots & 0 \\ 0 & -\lambda & \lambda & 0 & \cdots & 0 \\ 0 & 0 & -\lambda & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & -\lambda & \lambda \end{pmatrix}$$

- Other useful characterizations:

  $\{X(t), t \in T\}$ has integer-valued, right continuous sample paths.

I (*a*) For $t_0 < t_1 < \ldots < t_n$,
the random variable $X(t_0)$, $X(t_1)$-$X(t_0)$, $X(t_2)$-$X(t_1)$, …, $X(t_n)$-$X(t_{n-1})$ are independent,

and

(*b*) For $s < t$, $X(t)$-$X(s)$ ~ $\mathcal{P}\left(\lambda(t-s)\right)$;

i.e., $P\left[X(t) - X(s) = k\right] = \left(\lambda(t-s)\right)^k \dfrac{e^{-\lambda(t-s)}}{k!}$

II Let $T = [0, \infty)$ and $X(0) = 0$

Then $\{X(t), t \in T\}$ is Poisson with rate λ if and only if

(*a*) For $s < t$, given that $X(t)$-$X(s) = n$, the jump times of $\{X(\cdot)\}$ in $[s,t]$ are uniformly distributed over $\{t \in \mathbb{R}^n : s < \tau_1 < \tau_2 < \ldots < \tau_n < t\}$,

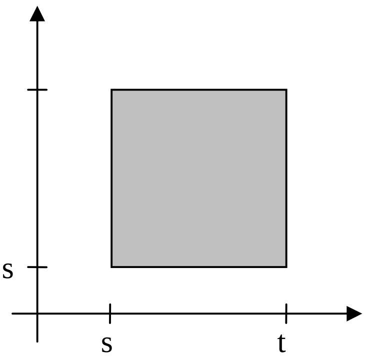
- For example,
  - If given number of jump = 2 during time s and t,

For each jump, the time that is occurs distributed according to

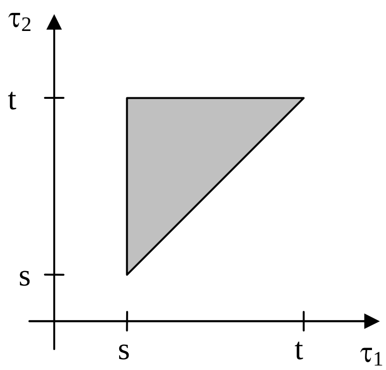
$f_J(u) = \frac{1}{s-t}$, $s \le u \le t$ (i.i.d.)

The following plot shows the joint-pdf of the times of jump:,

$f_{J,J'}(u,v) = \frac{1}{(s-t)^2}$, $s \le u,v \le t$.



- If also numbered the jump, then the second jump has to occur after the first jump: $f_{J_1,J_2}(\tau_1,\tau_2) = \frac{1}{\frac{1}{2}(s-t)^2}$, $s \le \tau_1 < \tau_2 \le t$.

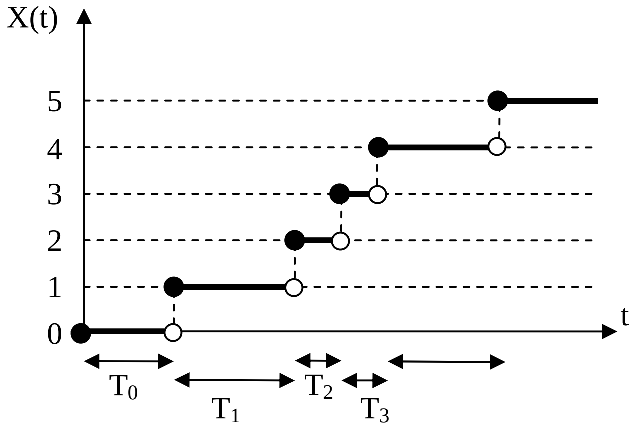


And (*b*) For $s < t$, $X(t)$-$X(s)$ ~ $\mathcal{P}(\lambda(t-s))$.

- Holding time T_k are i.i.d. $\mathcal{E}(\lambda)$ random variable.

  ET = $\frac{1}{\lambda}$

  All states are transient (in fact, non-return)



- Examples
  - Radioactive disintegrations (hence Geiger counter counts)
  - Arrival times of customers
  - Raindrop arrivals in a water glass.
  - Packet arrivals at a data communication network node.
- The initial condition for the Poisson process is $\boxed{p_0(0)=1}$, and $p_j(0)=0$ for $j > 0$. (At time 0, start at state 0).
- Derivation of the state probabilities from the transition rate matrix.

  From $\underline{p}'(t)=\underline{p}(t)Q$, we have

$$\left[p_0'(t), p_1'(t), \ldots\right] = \left[p_0(t), p_1(t), \ldots\right] \begin{pmatrix} -\lambda & \lambda & 0 & 0 & \cdots & 0 \\ 0 & -\lambda & \lambda & 0 & \cdots & 0 \\ 0 & 0 & -\lambda & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & -\lambda & \lambda \end{pmatrix}.$$

  Hence, $p_0'(t)=-\lambda p_0(t)$, and $p_j'(t)=-\lambda p_j(t)+\lambda p_{j-1}(t)$ for $j \ge 1$.

  From $p_0'(t)=-\lambda p_0(t)$, we have $p_0(t)=ce^{-\lambda t}$. The initial condition $p_0(0)=1$ requires that $c = 1$. Hence, $p_0(t)=e^{-\lambda t}$.

  We will show that $p_j(t)=\dfrac{(\lambda t)^j}{j!}e^{-\lambda t}$ by induction. We have already shown that this is true for the case $j = 1$. Now assume that $p_j(t)=\dfrac{(\lambda t)^j}{j!}e^{-\lambda t}$ for $j = 0, 1, \ldots,$ k-1. Now, from $p_k'(t)=-\lambda p_k(t)+\lambda p_{k-1}(t)=-\underbrace{\lambda}_{\alpha(t)} p_k(t)+\underbrace{\lambda\dfrac{(\lambda t)^{k-1}}{(k-1)!}e^{-\lambda t}}_{\beta(t)}$. The solution for this differential equation is $p_k(t) = e^{-\int \alpha(t)dt} \int \beta(t) e^{\int \alpha(t)dt} dt + ce^{-\int \alpha(t)dt}$.

Now, $\int \alpha(t)dt = \lambda t$. So we have

$$p_k(t) = e^{-\lambda t} \int \lambda \frac{(\lambda t)^{k-1}}{(k-1)!} \cancel{e^{-\lambda t} e^{\lambda t}} dt + ce^{-\lambda t} = e^{-\lambda t} \frac{\lambda^k t^k}{k!} + ce^{-\lambda t}.$$

Using the initial condition $p_k(0) = 0$, we have $c = 0$, and hence,

$$p_k(t) = e^{-\lambda t} \frac{\lambda^k t^k}{k!}.$$

- State diagram:

  At time t, consider the next dt time unit.

  - Pr[at least one jump occurs before dt] = Pr[1$^{\text{st}}$ jump occurs before dt]

    $$= 1 - e^{-\lambda dt} \frac{(\lambda dt)^0}{0!} = 1 - e^{-\lambda dt} = 1 - (1 - \lambda dt) + o(dt) = \lambda dt + o(dt).$$

    $$\lim_{dt \to 0} \frac{P\left[1^{\text{st}} \text{ jump before dt}\right]}{dt} = \lim_{dt \to 0} \frac{\lambda dt + o(dt)}{dt} = \lambda$$

  - $$\lim_{dt \to 0} \frac{P\left[1^{\text{st}} \text{ jump before dt}\right]}{dt} = \lim_{dt \to 0} \frac{1 - e^{-\lambda dt}}{dt} = \lim_{dt \to 0} \frac{\frac{d}{dt}\left(1 - e^{-\lambda dt}\right)}{\frac{d}{dt} dt} = \lim_{dt \to 0} \frac{\lambda e^{-\lambda dt}}{1} = \lambda$$

  - $P\left[2^{\text{nd}} \text{ jump before dt}\right]$

    $$= P\left[1^{\text{st}} \text{ jump before dt}\right] P\left[2^{\text{nd}} \text{ jump before dt} \middle| 1^{\text{st}} \text{ jump before dt}\right]$$

    We know that $P\left[2^{\text{nd}} \text{ jump before dt} \middle| 1^{\text{st}} \text{ jump before dt}\right] \le P\left[1^{\text{st}} \text{ jump before dt}\right]$ because the time interval is smaller.

    Thus,

    $$P\left[2^{\text{nd}} \text{ jump before dt}\right] \le \left(P\left[1^{\text{st}} \text{ jump before dt}\right]\right)^2 = \left(1 - e^{-\lambda dt}\right)^2 = (\lambda dt)^2$$

    $$\lim_{dt \to 0} \frac{P\left[2^{\text{nd}} \text{ jump before dt}\right]}{dt} = 0$$

$$\begin{tikzcd} 0 \arrow[r, bend left, "\lambda"] & 1 \arrow[r, bend left, "\lambda"] & 2 \arrow[r, bend left, "\lambda"] & 3 \arrow[r, bend left, "\lambda"] & {} \end{tikzcd}$$

  - This is consistent with what we have in the Q matrix.