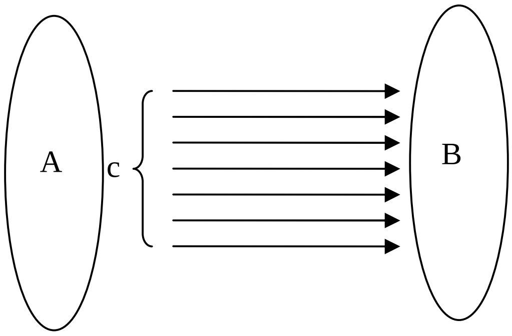
Engsett Model (Erlang-B)



- Description: "Blocked-calls lost model"
  - Consider a central exchange with ***k* users** (subscribers) sharing ***c* trunks** (trunks). When $k > c$, blocking occurs. This is the case of principal interest.
    Assume that the trunks are for long-distance calls to other exchanges, so none of the *k* users speaks over these trunks to any other of the *k* users.
  - Idle users each generate/initiate calls at rate λ independent, exponential.
    $\equiv$ idle users places next call attempt after $\mathcal{E}(\lambda)$ time passes
    $\equiv$ idle users activate $\mathcal{E}(\lambda)$
  - Busy users speak for $\mathcal{E}(\mu)$ durations, independently.
    $\equiv$ Busy users deactivate $\mathcal{E}(\mu)$
    $\equiv$ Active users each terminate calls at rate μ
  - Blocked calls = calls arriving when all *c* trunks are busy.
  - Blocked calls are "lost". Users will not try to generate new call attempts immediately when blocked (i.e., no retrial/redial). Rather, such an unserviced user simply returns to the pool of $k-c$ idle users who generate new requests for service at a combined rate of $(k-c)\lambda$.
- Let $\{X(t)\}$ = Number of calls in progress at time *t*.
  = Number of busy trunks at time *t*.

  $0 \le X(t) \le c$

  $k - X(t)$ = number of idle users/subscribers at time *t*.

  Under our assumptions, $\{X(t)\}$ is a time-continuous homogeneous Markov chain.
- *Q matrix derivation*:
  Suppose there are *i* active users at time *t*.
  $P(\text{all i of these are still active at time } t+dt)$

$$= P\left(1^{st} \text{ one is still active}\right) P\left(2^{nd} \text{ one is still active}\right) \times \ldots \times P\left(i^{th} \text{ one is still active}\right)$$

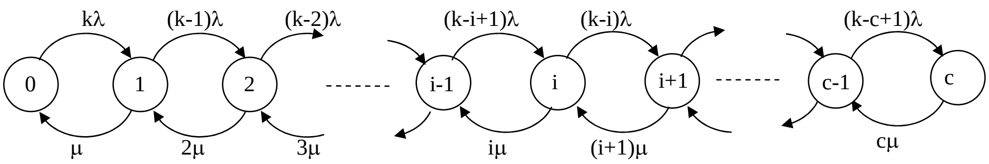
$$= \left(e^{-\mu dt}\right)^{i} = e^{-i\mu dt}$$

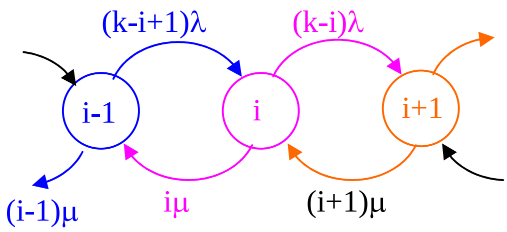
$$P\left(\text{At least one hangs up between t and dt}\right) = 1 - e^{-i\mu dt} \approx 1 - \left(1 - i\mu dt\right) = i\mu dt$$

$$\lim_{dt \to 0} \frac{P\left(\text{one } \left(\text{or more}\right) \text{ hangs up between t and dt}\right)}{dt} = i\mu$$

Therefore, $Q_{i,i-1} = i\mu$

Similar reasoning gives $Q_{i,i+1} = \left(k - i\right)\lambda$





$$\begin{array}{r} \\ \\ \text{i-1} \\ \text{i} \\ \text{i+1} \\ \\ \end{array} \begin{array}{ccccccccc} & & & \text{i-1} & \text{i} & \text{i+1} & & & \\ 0 & \ddots & \ddots & \ddots & 0 & 0 & 0 & 0 & 0 \\ \ldots & 0 & (i-1)\mu & -\left((i-1)\mu + (k-i+1)\lambda\right) & (k-i+1)\lambda & 0 & 0 & 0 & \ldots \\ \ldots & 0 & 0 & i\mu & -\left(i\mu + (k-i)\lambda\right) & (k-i)\lambda & 0 & 0 & \ldots \\ \ldots & 0 & 0 & 0 & (i+1)\mu & -\left((i+1)\mu + (k-i-1)\lambda\right) & (k-i-1)\lambda & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & \ddots & \ddots & \ddots & 0 \end{array}$$

- Underlying fact:
  If $T_1, T_2, \ldots, T_n$ are independent exponential random variable with $T_k \sim \mathcal{E}\left(\alpha_k\right)$, then, to first order in dt, the probability that the first of these exponential "clocks" to go off does so in $[0, dt]$ is $\left(\alpha_1 + \alpha_2 + \ldots + \alpha_n\right) dt$.

$$\Pr\left[1^{st} \text{ one is before } dt\right] = \Pr\left[\text{At least one before } dt\right]$$

$$= 1 - \Pr\left[T_k > dt, 1 \le k \le n\right]$$

$$= 1 - \prod_{k=1}^{n} P\left(T_k > dt\right) = 1 - \prod_{k=1}^{n} e^{-\alpha_k dt} = 1 - e^{-\left(\sum_{k=1}^{n} \alpha_k\right) dt}$$

$$= 1 - \left(1 - \left(\sum_{k=1}^{n} \alpha_k\right) dt + o\left(dt\right)\right) = \left(\sum_{k=1}^{n} \alpha_k\right) dt + o\left(dt\right)$$

- Dynamic equations: $\underline{p}'\left(t\right) = \underline{p}\left(t\right) Q\left(t\right)$ where $p_i\left(t\right) = \Pr\left[X\left(t\right) = i\right]$. This is

$$\left(\frac{dp_0(t)}{dt},\ldots,\frac{dp_i(t)}{dt},\ldots,\frac{dp_c(t)}{dt}\right)$$

$$=\left(p_0(t),\ldots,p_i(t),\ldots,p_c(t)\right)\begin{pmatrix} \ddots & 0 & \iddots \\ \cdots & (k-i+1)\lambda & \cdots \\ \cdots & -\left(i\mu+(k-i)\lambda\right) & \cdots \\ \cdots & (i+1)\mu & \cdots \\ \iddots & 0 & \ddots \end{pmatrix}.$$

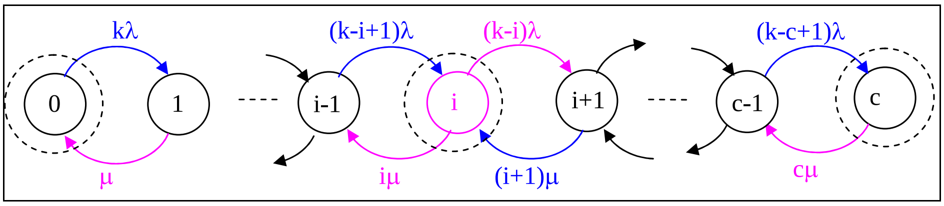
Or, equivalently,

$$\frac{dp_i(t)}{dt}=(k-i+1)\lambda p_{i-1}(t)-\left(i\mu+(k-i)\lambda\right)p_i(t)+(i+1)\mu p_{i+1}(t)\ ;\ 1\le i\le c\text{-}1$$

Easier to get from this from $\frac{d}{dt}p_j(t)=\sum_{i\neq j}p_i(t)q_{i,j}(t)-\sum_{k\neq j}p_j(t)q_{j,k}(s)$ or

$$\begin{pmatrix}\text{Instantaneous rate of}\\ \text{change of probability}\\ \text{of state j}\end{pmatrix}=\begin{pmatrix}\text{Instantaneous flow of}\\ \text{probability into state j}\end{pmatrix}-\begin{pmatrix}\text{Instantaneous flow of}\\ \text{probability out of state j}\end{pmatrix},$$ and

the diagram.



Either way, we have:

$$\begin{cases}\frac{dp_0(t)}{dt}=k\lambda p_0(t)-\lambda\mu p_1(t)\\ \frac{dp_i(t)}{dt}=(k-i+1)\lambda p_{i-1}(t)-\left(i\mu+(k-i)\lambda\right)p_i(t)+(i+1)\mu p_{i+1}(t) \quad 1\le i\le c-1\\ \frac{dp_c(t)}{dt}=(k-c+1)\lambda p_{c-1}(t)-c\mu p_c(t)\end{cases}$$

- Equilibrium distribution: Truncated binomial: $p_i=\frac{\binom{k}{i}\rho^i}{\sum_{j=0}^{c}\binom{k}{j}\rho^j},\ \rho:=\frac{\lambda}{\mu}$

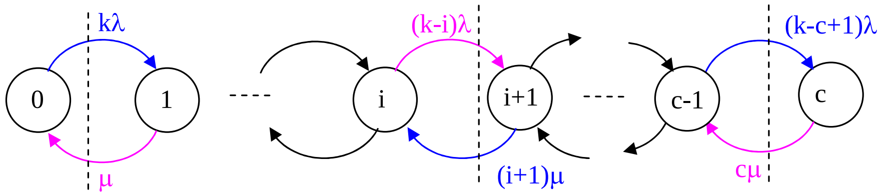
- Set $\frac{dp_i(t)}{dt}=0,\forall i,\ 0\le i\le c$

  This will give

$$\begin{cases} k\lambda p_0 = \mu p_1 \\ \left(i\mu + (k-i)\lambda\right) p_i = (k-i+1)\lambda p_{i-1} + (i+1)\mu p_{i+1} \quad 1 \le i \le c-1 \\ c\mu p_c = (k-c+1)\lambda p_{c-1} \end{cases}$$

Use c-1 equations of these c+1 equations plus $\sum_{k=0}^{c} p_i = 1$.

- From ⋮ partitioning, we get



$$(k-i)\lambda p_i = (i+1)\mu p_{i+1} \Rightarrow p_{i+1} = \frac{k-i}{i+1}\frac{\lambda}{\mu} p_i = \frac{k-i}{i+1}\rho p_i \;;\; 0 \le \mathrm{i} \le \mathrm{c\text{-}1}$$

$$p_i = \frac{k-i-1}{i}\rho p_{i-1} = \frac{\left(k-(i-1)\right)\left(k-(i-2)\right)}{i(i-1)}\rho^2 p_{i-2}$$

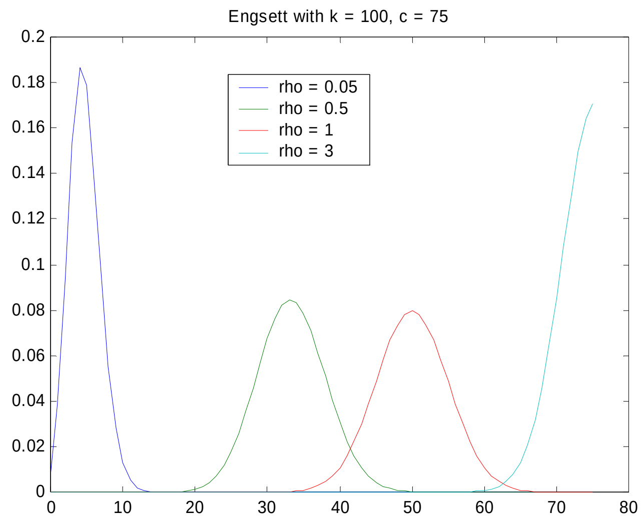
$$= \frac{\left(k-(i-1)\right)\left(k-(i-2)\right)\times\ldots\times(k)}{i(i-1)\times\ldots\times 1}\rho^c p_0$$

$$= \frac{\frac{k!}{(k-i)!}}{i!}\rho^i p_0 = \frac{k!}{i!(k-i)!}\rho^i p_0 = \binom{k}{i}\rho^i p_0$$

$$\sum_{i=0}^{c} p_i = 1 \Rightarrow \sum_{i=0}^{c}\binom{k}{i}\rho^i p_0 = 1 \Rightarrow p_0 = \frac{1}{\sum_{i=0}^{c}\binom{k}{i}\rho^i}$$

$$p_i = \frac{\binom{k}{i}\rho^i}{\sum_{j=0}^{c}\binom{k}{j}\rho^j} \;;\; 0 \le i \le c$$

- For small values of ρ, this distribution is slanted heavily toward the small values of i.
  It's unimodal for intermediate values of ρ, and
  it's slated heavily toward the large values of i for large values of ρ.



- Def: $\boldsymbol{P_b} = \Pr[\text{call attempt is blocked}]$

  $= \Pr[\text{get a busy signal}]$

  $= \Pr[\text{long-term fraction of all the call attempts that get blocked}]$

- $$P_b = \frac{(k-c)\binom{k}{c}\rho^c}{\sum_{j=0}^{c}(k-j)\binom{k}{j}\rho^j}$$

Proof. Since the Engsett arrival process is state-dependent and hence not Poisson, Engsett arrivals do not see steady-state conditions, so the blocking probability is not equal to p_c.

We may assume that steady-state conditions prevail; they will in the long run regardless of the value of ρ because we have an irreducible, finite-state time-continuous chain.

In the steady state, the value of p_i represents the fraction of the time axis during which the system is in state i, or equivalently the probability that the system is in state i at a "randomly chosen instant."

However, the density of calling attempts in the Engsett model varies with the state of the system. When in state i, call attempts occur at rate $(k\text{-}i)\lambda$.

The fraction of all call attempts that occur when the system is in state i is not p_i but rather $f_i = \dfrac{(k-i)\cancel{\lambda} p_i}{\sum_{j=0}^{c}(k-j)\cancel{\lambda} p_j} = \dfrac{(k-i)\,p_i}{\sum_{j=0}^{c}(k-j)\,p_j}$.

Since call attempts get blocked if and only if they occur when the system is in state c, the blocking probability is

$$P_b = f_c = \frac{(k-c)\,p_c}{\sum_{j=0}^{c}(k-j)\,p_j} = \frac{(k-c)\frac{\binom{k}{c}\rho^c \cancel{p_0}}{\cancel{\sum_{i=0}^{c}\binom{k}{i}\rho^i}}}{\sum_{j=0}^{c}(k-j)\frac{\binom{k}{j}\rho^j \cancel{p_0}}{\cancel{\sum_{i=0}^{c}\binom{k}{i}\rho^i}}} = \frac{(k-c)\binom{k}{c}\rho^c}{\sum_{j=0}^{c}(k-j)\binom{k}{j}\rho^j}$$

- $\boxed{\lim_{\rho\to\infty} P_b = 1,\ \lim_{\rho\to 0} P_b = 0,\ \lim_{\frac{k}{c}\to\infty} P_b = 1}$

Proof. Since the term ρ^c has the highest order in $(k-c)\binom{k}{c}\rho^c$ and

$$\sum_{j=0}^{c}(k-j)\binom{k}{j}\rho^j,\ \lim_{\rho\to\infty}\frac{(k-c)\binom{k}{c}\rho^c}{\sum_{j=0}^{c}(k-j)\binom{k}{j}\rho^j} = \frac{(k-c)\binom{k}{c}\rho^c}{(k-c)\binom{k}{c}\rho^c} = 1$$

Proof. $\lim_{\rho\to 0} P_b = \lim_{\rho\to 0}\frac{(k-c)\binom{k}{c}\rho^c}{\sum_{j=0}^{c}(k-j)\binom{k}{j}\rho^j} = \frac{0}{k} = 0$

Proof. For $j < c < k$

$$\frac{(k-j)\binom{k}{j}\rho^j}{(k-c)\binom{k}{c}\rho^c} = \frac{k-j}{k-c}\frac{\frac{k!}{j!(k-j)!}}{\frac{k!}{c!(k-c)!}}\rho^{j-c} = \frac{k-j}{k-c}\frac{c!(k-c)!}{j!(k-j)!}\rho^{j-c}$$

$$= \frac{k-j}{k-c}\frac{c(c-1)\cdots(j+1)}{(k-j)(k-(j+1))\cdots(k-(c-1))}\rho^{j-c}$$

$$= \frac{c(c-1)\cdots(j+1)}{(k-(j+1))\cdots(k-(c-1))(k-c)}\rho^{j-c}$$

$$\le \frac{c^{c-j}}{(k-c)^{c-j}}\rho^{j-c} = \frac{1}{\left(\rho\frac{k}{c}-\rho\right)^{c-j}} \xrightarrow{\frac{k}{c}\to\infty} 0$$

Hence, $\lim_{\frac{k}{c}\to\infty} P_b = \lim_{\frac{k}{c}\to\infty} \dfrac{(k-c)\binom{k}{c}\rho^c}{\sum_{j=0}^{c}(k-j)\binom{k}{j}\rho^j} = \dfrac{1}{\sum_{j=0}^{c}\lim_{\frac{k}{c}\to\infty}\dfrac{(k-j)\binom{k}{j}\rho^j}{(k-c)\binom{k}{c}\rho^c}}$

$$= \frac{1}{0+0+\cdots+0+1} = 1$$

- $P_b = 0$ for $k = c$.
- If there is only one fewer trunk than subscriber ($c = k$-1), then $P_b = \left(\dfrac{\rho}{1+\rho}\right)^c$

Proof. Note that $\sum_{j=1}^{k} j\binom{k}{j}\rho^j = \sum_{j=0}^{k} j\binom{k}{j}\rho^j = k\rho(1+\rho)^{k-1}$. Hence,

$$\begin{aligned}\sum_{j=0}^{k}(k-j)\binom{k}{j}\rho^j &= \left(k\sum_{j=0}^{k}\binom{k}{j}\rho^j\right) - \left(\sum_{j=0}^{k} j\binom{k}{j}\rho^j\right)\\ &= k(1+\rho)^k - k\rho(1+\rho)^{k-1}\\ &= k(1+\rho)^k\left(1-\frac{\rho}{1+\rho}\right) = k(1+\rho)^{k-1}\end{aligned}$$

Because when $j = k$, $(k-j)\binom{k}{j}\rho^j = 0$, we have

$$\sum_{j=0}^{k-1}(k-j)\binom{k}{j}\rho^j = \sum_{j=0}^{k}(k-j)\binom{k}{j}\rho^j = k(1+\rho)^{k-1}.$$

$$P_b = \frac{((c+1)-c)\binom{c+1}{c}\rho^c}{\sum_{j=0}^{c}((c+1)-j)\binom{c+1}{j}\rho^j} = \frac{(c+1)\rho^c}{(c+1)(1+\rho)^c} = \left(\frac{\rho}{1+\rho}\right)^c$$

- $P_b < p_c$ for $c < k$

Proof. We want to compare $P_b = \dfrac{(k-c)\binom{k}{c}\rho^c}{\sum_{j=0}^{c}(k-j)\binom{k}{j}\rho^j}$ and $p_c = \dfrac{\binom{k}{c}\rho^c}{\sum_{j=0}^{c}\binom{k}{j}\rho^j}$

$$k-c \le k-j$$

$$\sum_{j=0}^{c}(k-c)\binom{k}{j}\rho^j < \sum_{j=0}^{c}(k-j)\binom{k}{j}\rho^j$$

$$(k-c)\sum_{j=0}^{c}\binom{k}{j}\rho^j < \sum_{j=0}^{c}(k-j)\binom{k}{j}\rho^j$$

$$\frac{(k-c)}{\sum_{j=0}^{c}(k-j)\binom{k}{j}\rho^j} < \frac{1}{\sum_{j=0}^{c}\binom{k}{j}\rho^j}$$

$$\frac{(k-c)\binom{k}{c}\rho^c}{\sum_{j=0}^{c}(k-j)\binom{k}{j}\rho^j} < \frac{\binom{k}{c}\rho^c}{\sum_{j=0}^{c}\binom{k}{j}\rho^j}$$

$$P_b < p_c$$

- Caution! $P_b \ne \Pr[X(t) = c$ and next pick-up precedes next hang-up$]$

  $\Pr[X(t) = c$ and next pick-up precedes next hang-up$]$

  $= P_c \Pr[U < V \mid$ in state $c]$

  where $U \sim \mathcal{E}((k-c)\lambda)$, $V \sim \mathcal{E}(c\mu)$, and $U \perp\!\!\!\perp V$

  $$= p_c\left(\frac{(k-c)\lambda}{(k-c)\lambda + c\mu}\right) = p_c\left(\frac{(k-c)\rho}{(k-c)\rho + c}\right)$$

Erlang Model

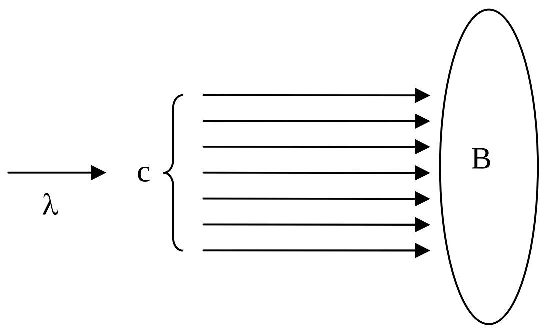
- Definition

  "Infinite" population of users initiates calls at a **combined** Poisson rate λ regardless of how many calls ($\le c$) are in progress

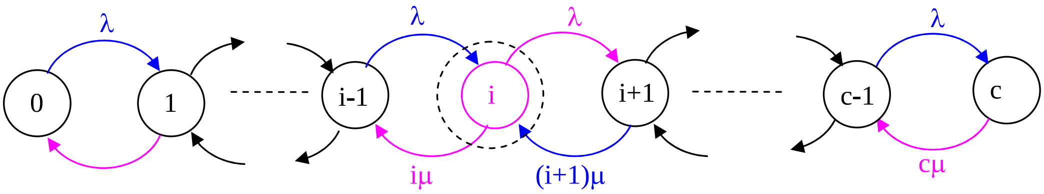
  Again

  Blocked calls lost.

  Holding times are i.i.d. $\mathcal{E}(\mu)$



- State Diagram:

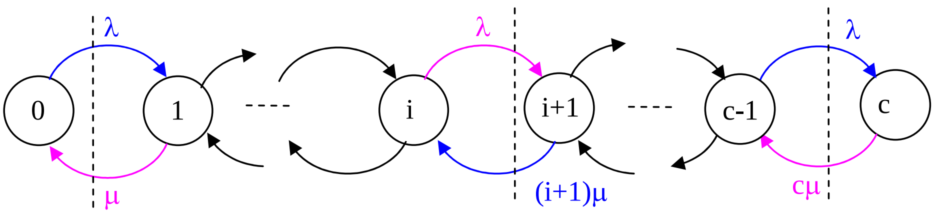


$$\begin{cases} p_0' = \mu p_1 - \lambda p_0 \\ p_i' = \lambda p_{i-1} + (i+1)\mu p_{i+1} - (\lambda + i\mu) p_i \;;1 \le i \le c-1 \\ p_c' = \lambda p_{c-1} - c\mu p_i \end{cases}$$

- $\boxed{p_i = \dfrac{\frac{\rho^i}{i!}}{\sum_{k=0}^{c} \frac{\rho^k}{k!}}}$ $0 \le i \le c$ (Truncated Poisson).

Proof 1.

Vertical dashed line give



$$\lambda p_0 = \mu p_1 \Rightarrow p_1 = \frac{\lambda}{\mu} p_0 = \rho p_0$$

$$\lambda p_i = (i+1)\mu p_{i+1} \Rightarrow p_{i+1} = \frac{1}{i+1}\frac{\lambda}{\mu} p_i = \frac{1}{i+1}\rho p_i$$

Hence, $p_i = \dfrac{\rho^i}{i!} p_0$.

From $\sum_{i=0}^{c} p_i = 1$, $p_0 \sum_{i=0}^{c} \dfrac{\rho^i}{i!} = 1 \Rightarrow p_0 = \dfrac{1}{\sum_{i=0}^{c} \frac{\rho^i}{i!}}$

Thus, $p_i = \dfrac{\frac{\rho^i}{i!}}{\sum_{k=0}^{c} \frac{\rho^k}{k!}}$ $0 \le \mathrm{i} \le \mathrm{c} \Rightarrow$ truncated Poisson

Proof 2.

Use formula $p_i = \frac{R_i}{\sum_{j=0}^{c} R_j}$, where $R_0 = 1$, $R_j = r_j r_{j-1} \cdots r_1$, $r_j = \frac{\lambda_{j-1}}{\mu_j} = \frac{\lambda}{j\mu} = \frac{\rho}{j}$. This implies $R_j = \frac{\rho^j}{j!}$.

- $$\boxed{P_b = p_c = \frac{\frac{\rho^c}{c!}}{\sum_{k=0}^{c} \frac{\rho^k}{k!}}}$$

Proof 1.

$$P_b = f_c = \frac{\cancel{\lambda} p_c}{\sum_{j=0}^{c} \cancel{\lambda} p_j} = \frac{p_c}{\sum_{j=0}^{c} p_j} = \frac{p_c}{1} = p_c$$

Proof 2. Because $\forall i \;\; \lambda_i = \lambda$, we already know that $P_b = f_c = p_c$.