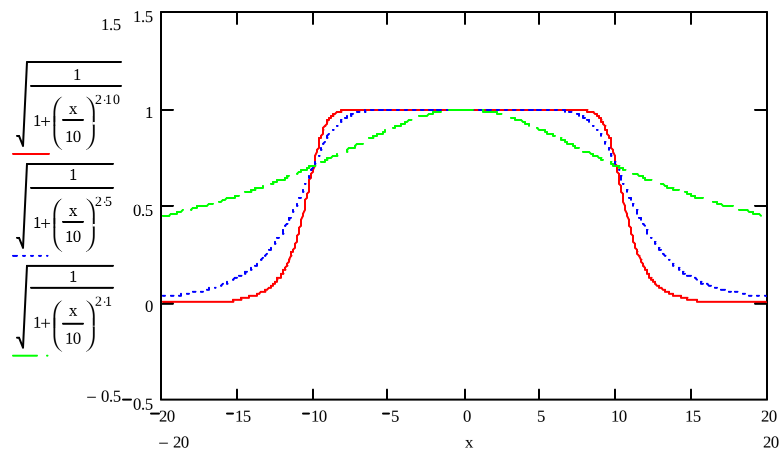
Continuous-time low-pass filter

Butterworth filter

- N^{th}-order Butterworth filter with cutoff frequency Ω_c :

  $$\left|\hat{H}_c(\Omega)\right|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

- Very flat @ 0 : (N-1) derivative = 0 @ 0



- Magnitude is monotonically decreasing in $|\Omega|$
- Passband error is worse near Ω_c
- $$\left|\hat{H}_c(\Omega_c)\right| = \sqrt{\frac{1}{1+\left(\frac{\Omega_c}{\Omega_c}\right)^{2N}}} = \sqrt{\frac{1}{2}} = 0.707$$

- $$H_c(s) = \frac{\Omega_c^N}{\prod_{i=1}^{N}(s-s_i)}$$
  - s_i : poles of $H_c(s)$
    - Odd N: $\Omega_c \times \left(e^{jm\frac{p}{N}} \text{ that are in } \mathrm{Re}\{s\}<0 \right)$; $0 \le m < 2N$
    - Even N: $\Omega_c \times \left(e^{j\left(\frac{p}{2N}+m\frac{p}{N}\right)} \text{ that are in } \mathrm{Re}\{s\}<0 \right)$; $0 \le m < 2N$
  - ROC: region to the right of all poles

Fact: $\left|\hat{H}_c(\Omega)\right|^2 = H_c(-s)H_c(s)\big|_{s=j\Omega}$

- $H_c(s) = \dfrac{\Omega_c^N}{\prod_{i=1}^{N}(s-s_i)}$

$$H_c(-s)H_c(s) = \left(\frac{\Omega_c^N}{\prod_{i=1}^{N}(-s-s_i)}\right)\left(\frac{\Omega_c^N}{\prod_{i=1}^{N}(s-s_i)}\right) = \frac{\Omega_c^{2N}}{\left(\prod_{i=1}^{N}(-s-s_i)\right)\left(\prod_{i=1}^{N}(s-s_i)\right)}$$

From this equation, $H_c(-s)H_c(s)$ has 2N pole N s_i's and N -s_i's

Want $H_c(s)$ to be stable

$\Rightarrow$ have poles in $\mathrm{Re}\{s_i\} < 0$

$\Rightarrow$ the left-half-plane poles is for $H_c(s)$;

(the right-half-plane poles is for $H_c(-s)$)

- From $\left|\hat{H}_c(\Omega)\right|^2 = H_c(-s)H_c(s)\Big|_{s=j\Omega} \Rightarrow H_c(-s)H_c(s) = \left|\hat{H}_c(\Omega)\right|^2\Big|_{\Omega=\frac{s}{j}}$

$$H_c(-s)H_c(s) = \left|\hat{H}_c(\Omega)\right|^2\Big|_{\Omega=\frac{s}{j}} = \frac{1}{1+\left(\frac{s}{j\Omega_c}\right)^{2N}} = \frac{1}{1+\frac{s^{2N}}{j^{2N}\Omega_c^{2N}}}$$

$$= \frac{1}{1+\frac{s^{2N}}{(-1)^N\Omega_c^{2N}}} = \frac{\Omega_c^{2N}}{\Omega_c^{2N}+(-1)^N s^{2N}}$$

From this equation, $H_c(-s)H_c(s)$ has 2N pole at

$$\Omega_c^{2N}+(-1)^N s_i^{2N} = 0 \Rightarrow s_i^{2N} = -(-1)^N\Omega_c^{2N}$$

- If N is odd, poles: $s_i^{2N} = \Omega_c^{2N}$

  $s_i = \Omega_c \times \left((2N)^{th}\text{-roots of unity}\right) = \Omega_c \times e^{jm\frac{2p}{2N}}$; $0 \le m < 2N$

- If N is even, poles: $s_i^{2N} = -\Omega_c^{2N} = e^{jp}\Omega_c^{2N}$

  $s_i = e^{j\left(\frac{p}{2N}+m\frac{2p}{2N}\right)}\Omega_c$

Chebyshev filter

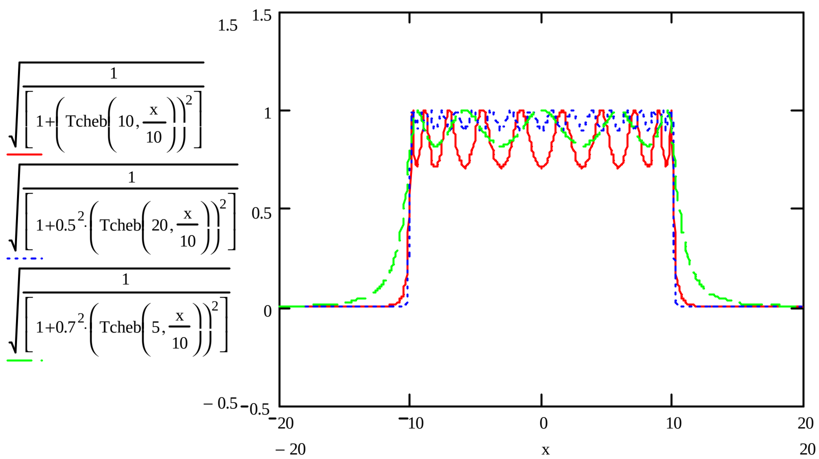
Type I

- N^{th}-order Chebyshev filter with cutoff Ω_c type I

$$\left|\hat{H}_c(\Omega)\right|^2 = \frac{1}{1+e^2 V_N^2\left(\frac{\Omega}{\Omega_c}\right)}$$

- $V_N(x)$ = the N^{th} Chebyshev polynomial in x
- $V_{N+1}(x) = 2xV_N(x) - V_{N-1}(x)$
- $\left|\hat{H}_c(\Omega_c)\right|^2 = \frac{1}{1+e^2 V_N^2\left(\frac{\Omega_c}{\Omega_c}\right)} = \frac{1}{1+e^2} \approx 1$

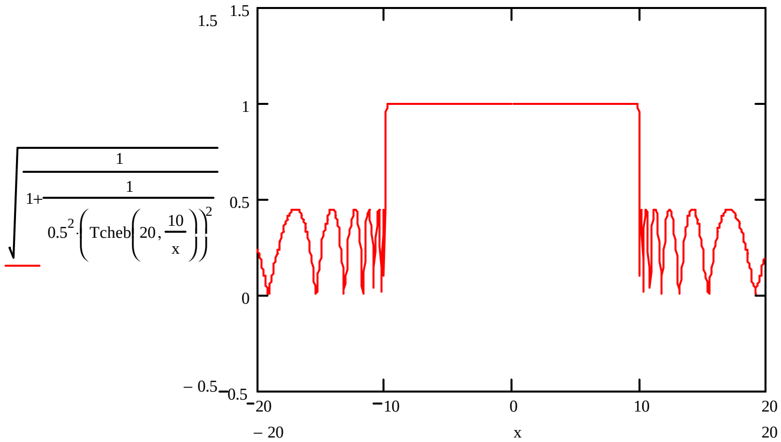
- Euiripple in passband ⇒ error is distributed uniformly
  - To achieve a given passband max. error, require lower-order (N) than butterworth
- Monotonic in stopband



Type II

- N^{th}-order Chebyshev filter with cutoff Ω_c type II

$$\left|\hat{H}_c(\Omega)\right|^2 = \frac{1}{1+\frac{1}{e^2 V_N^2\left(\frac{\Omega_c}{\Omega}\right)}}$$



- Monotonic in passband
- Equiripple in stopband

Elliptic filter

- N^{th}-order Elliptic filter with cutoff Ω_c :

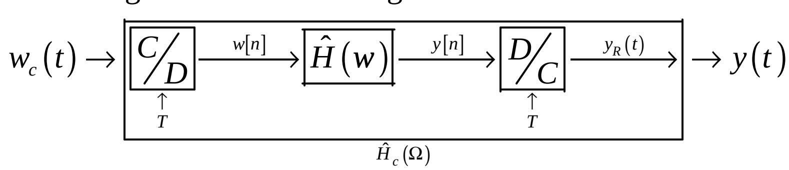
$$\left|\hat{H}_c(\Omega)\right|^2 = \frac{1}{1+e^2 U_N^2\left(\frac{\Omega}{\Omega_c}\right)}$$

  - $U_N(x) = N^{th}$ Jacobian elliptic function of x
- Equiripple both in passband and stopband
- Require smaller N than Chebychev to achieve max error in passband or stopband

Digital filter design

Old-fashioned DSP paradigm

- Processing continuous-time signals with the aid of discrete-time systems



Objective: Given $\hat{H}_{desired}(\Omega) = \hat{H}_c(\Omega)$, find T and $\hat{H}(w)$ so that

$\hat{Y}(\Omega) \approx \hat{H}_c(\Omega)\hat{W}_c(\Omega)$

Restrict to band-limited input $w_c(t)$ and $T < \frac{p}{\Omega_m}$

Solution: use $\hat{H}(w) = \hat{H}_c\left(\frac{w}{T}\right)$, $|w| \le p$

- Note: here, we are not sampling $h_c(t)$ to get $h[n]$. So, can't find the relationship using the deconstruction. $h[n]$ is some signal that, when used, will make the whole discrete system $\left(\boxed{C/D}, \boxed{\hat{H}(w)}, \boxed{D/C}\right)$ act like $\hat{H}_c(\Omega)$.

  Proof

  - $\hat{W}(\Omega T) = \frac{1}{T}\hat{W}_c(\Omega)$ for $|\Omega| \le \frac{p}{T}$, assuming no aliasing

    $$\hat{W}(w) = \sum_{k=-\infty}^{\infty}\left(\frac{1}{T}\hat{W}_c\left(\frac{w}{T} + k\frac{2p}{T}\right)\right)$$

    If no aliasing, $\hat{W}(w) = \frac{1}{T}\hat{W}_c\left(\frac{w}{T}\right)$; for $-p \le w \le p$

    or $\hat{W}(w)\, p_p(w) = \frac{1}{T}\hat{W}_c\left(\frac{w}{T}\right)$

  - $\hat{W}(\Omega T) = \sum_{k=-\infty}^{\infty}\frac{1}{T}\hat{W}_c\left(\Omega + k\frac{2p}{T}\right)$

    $= \frac{1}{T}\hat{W}_c(\Omega)$ for $|\Omega| \le \frac{p}{T}$, assuming no aliasing

  - $\hat{Y}_R(\Omega) = \hat{H}(\Omega T)\hat{W}_c(\Omega)$,$\forall w$

    From $\hat{Y}(w) = \hat{H}(w)\hat{W}(w)$, and $\hat{Y}_R(\Omega) = T\left(\hat{Y}(w)\, p_p(w)\right)\Big|_{w=\Omega T}$,

    $\hat{Y}_R(\Omega) = T\left(\hat{H}(w)\,\hat{W}(w)\, p_p(w)\right)\Big|_{w=\Omega T}$.

    Substitute $\hat{W}(w)\, p_p(w) = \frac{1}{T}\hat{W}_c\left(\frac{w}{T}\right)$:

    $$\hat{Y}_R(\Omega) = T\left(\hat{H}(w)\frac{1}{T}\hat{W}_c\left(\frac{w}{T}\right)\right)\Bigg|_{w=\Omega T} = \hat{H}(\Omega T)\hat{W}_c(\Omega)$$

  - Thus, want $\hat{H}(w) = \hat{H}_c\left(\frac{w}{T}\right)$, $|w| \le p$, or

    $\hat{H}(\Omega T) = \hat{H}_C(\Omega)$ at least for $|\Omega| \le \frac{p}{T}$.

    Don't need to worry about $|\Omega| > \frac{p}{T}$ since $\hat{W}_c(\Omega)$ = 0 there.

New-fashioned DSP paradigm

- Given $\hat{H}_{des}(\omega)$.

  Design an implementable discrete-time system with $\hat{H}(\omega) \approx \hat{H}_{des}(\omega)$

IIR filter design

- IIR ⇒ infinite-duration impulse response
- Always assume h[n] is real-valued

IIR filter design using impulse invariance

- Design a discrete-time low-pass filter $\hat{H}_{des}(w)$

  (from available $\hat{H}_c(\Omega)$)

  with cutoff w_c

  need to meet design specs
- Pick $T_d > 0$; d = design
- Via w = W$\mathbf{T_d}$, translate $\hat{H}_{des}(w) \to \hat{H}_{c,des}(\Omega)$; $\left(\Omega_c = \frac{w_c}{T_d}\right)$ along with design specs
- Design $h_c(t)$ that meets the continuous-time design specs and
  - **stable & causal**
  - rational $H_c(s)$
- Set $h[n] = T_d h_c(nT_d)$. See whether this work.

- $\hat{H}(w) = T_d \sum_{k=-\infty}^{\infty} \frac{1}{T_d} \hat{H}_c\left(\frac{w}{T_d} + k\frac{2p}{T_d}\right) = \sum_{k=-\infty}^{\infty} \hat{H}_c\left(\frac{w}{T_d} + k\frac{2p}{T_d}\right)$
- $T_d > 0$ doesn't matter
- $\frac{p}{T_d} > \Omega_c = \frac{w_c}{T_d}$ ⇒ small aliasing if $\hat{H}_c(\Omega) \approx 0$ outside some bound (Ω_c)
- Aliasing might cause $\hat{H}(w)$ not to meet original design specs (especially if $\hat{H}_c(\Omega)$ barely does the job in continuous time) ⇒ If this occurs, then should over-design

- $$\left\{\begin{array}{l} h_c(t) = \sum_\ell k_\ell e^{s_\ell t} u(t) \\ H_c(s) = \sum_\ell \frac{k_\ell}{s - s_\ell} \text{ ; } \operatorname{Re}\{s\} > \max_\ell \operatorname{Re}\{s_\ell\} \end{array}\right\} \to \left\{\begin{array}{l} h[n] = T_d \sum_\ell k_\ell (z_\ell)^n u[n] \text{ ; } z_\ell = e^{s_\ell T_d} \\ H(z) = T_d \sum_\ell \frac{k_\ell z}{z - z_\ell} \text{ ; } |z| > \max |z_\ell| \end{array}\right\}$$

  - $h_c(t)$ is stable, causal, and has rational $H_c(s)$ ⇒

    $h[n]$ is stable, causal, and has rational $H(z)$
  - $h_c(t)$ is not band-limited ⇒ there is aliasing when convert to $h[n]$
  - pole $H_c(s)$ @ s_0 ⇒ pole $H(z)$ @ $e^{s_0 T_d}$
  - $h_c(t) = \sum_\ell k_\ell e^{s_\ell t} u(t) \overset{\mathcal{L}}{\rightleftharpoons} H_c(s) = \sum_\ell \frac{k_\ell}{s - s_\ell}$; $\operatorname{Re}\{s\} > \max_\ell \operatorname{Re}\{s_\ell\}$
    - s_ℓ = poles of $H_c(s)$

- Stable iff all poles of rational *H(s)* lies in $\text{Re}\{s\} < 0$. So need $\text{Re}\{s_\ell\} < 0, \forall \ell$
- $h[n] = T_d h_c(nT_d) = T_d \sum_\ell k_\ell e^{s_\ell n T_d} u[n] = T_d \sum_\ell k_\ell \left(e^{s_\ell T_d}\right)^n u[n] = T_d \sum_\ell k_\ell (z_\ell)^n u[n]$
  - z_ℓ = poles of H(z) = $e^{s_\ell T_d}$
  - $h[n] = T_d \sum_\ell k_\ell (z_\ell)^n u[n] \overset{Z}{\rightleftharpoons} H(z) = T_d \sum_\ell \frac{k_\ell z}{z - z_\ell} = T_d \sum_\ell \frac{k_\ell z}{z - e^{s_\ell T_d}}$;

    $|z| > \max|z_\ell| = \max\left|e^{s_\ell T_d}\right|$
  - $\text{Re}\{s_\ell\} < 0 \Rightarrow |z_\ell| = \left|e^{s_\ell T_d}\right| < 1 \Rightarrow$ stable
- Reverse process is not uniquely determined

IIR filter design using bilinear transformation

- Want a discrete-time low-pass filter $\hat{H}_{des}(w)$ with cutoff ω_c

  need to meet design specs
- Pick any $T_d > 0$
- Via $\Omega = \frac{2}{T_d}\tan\left(\frac{w}{2}\right)$, translate $\hat{H}_{des}(w) \to \hat{H}_{c,des}(\Omega)$ (equal height); along with design specs
- Design $h_c(t)$ / $H_c(s)$ that meets the continuous-time design specs and
  - stable & causal
  - rational $H_c(s)$
- $H(z) = H_c\left(s = \frac{2}{T_d}\frac{1-z^{-1}}{1+z^{-1}}\right)$

  See whether $\hat{H}(w) = H\left(z = e^{jw}\right)$ work.

- Idea

  Trapezoidal approximation

  $$y(t) = \int w(t)\,dt \;\Rightarrow\; y(nT_d) - y((n-1)T_d) \approx \frac{T_d}{2}\Big(w(nT_d) + w((n-1)T_d)\Big)$$

  $$y[n] - y[n-1] \approx \frac{T_d}{2}\big(w[n] + w[n-1]\big)$$

  $$\left(1 - z^{-1}\right)Y(z) \approx \frac{T_d}{2}\left(1 + z^{-1}\right)W(z)$$

  $$H(z) = \frac{Y(z)}{W(z)} \approx \frac{T_d}{2}\frac{1+z^{-1}}{1-z^{-1}}$$

  Imaginary axis in s-space $(s = j\Omega)$

  maps onto

unit circle in z-space $\left(z = e^{jw}\right)$

$$\Rightarrow\; j\Omega = \frac{2}{T_d}\frac{1-e^{-jw}}{1+e^{-jw}} \;\Rightarrow$$

$$\Omega = \frac{2}{jT_d}\frac{1-e^{-jw}}{1+e^{-jw}} = \frac{2}{jT_d}\frac{e^{-j\frac{w}{2}}}{e^{-j\frac{w}{2}}}\frac{e^{j\frac{w}{2}}-e^{-j\frac{w}{2}}}{e^{-j\frac{w}{2}}+e^{-j\frac{w}{2}}} = \frac{2}{jT_d}\frac{2j\sin\left(\frac{w}{2}\right)}{2\cos\left(\frac{w}{2}\right)} = \frac{2}{T_d}\tan\left(\frac{w}{2}\right)$$

- Note: Continuous-time integrator ⇒ $H_I(s) = \frac{1}{s}$

- All of Ω-space, ie, -∞ < Ω < ∞

  maps onto

-π ≤ ω ≤ π in ω-space (and 2π-periodic)

- No aliasing
- Non-linear mapping between ω and Ω
  - Not a big problem if $\hat{H}_{c,des}(\Omega) \approx$ piecewise-constant
- $\hat{H}_{c,des}(\Omega)$'s phase characteristics get dangerously twisted
- s_0 is a pole of $H_c(s) \Rightarrow z_0 = \dfrac{\frac{2}{T_d}+s_0}{\frac{2}{T_d}-s_0}$ is a pole of $H(z)$
- If $\mathrm{Re}\{S_0\} < 0$, then $|z_0| < 1$
- $H(z)$ is rational and stable, if $H_c(s)$ is rational, causal, and stable

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- Let $H(z) = H_c\left(s = b\,\dfrac{1-z^{-M}}{1+z^{-M}}\right)$,

  $H_c(s)$ is rational & stable $\left(\mathrm{Re}\{s_0\} < 0\right)$, b is real, and M is a non-zero integer.

  Then
  - $H(z)$ is rational
  - *H(z)* is stable $\left(|z_0| < 1\right)$ if β and M have same sign

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Proof

$$s_0 = b\,\frac{1-z_0^{-M}}{1+z_0^{-M}} \Rightarrow s_0 + s_0 z_0^{-M} = b - b\,z_0^{-M} \Rightarrow z_0^{-M} = \frac{b-s_0}{b+s_0}$$

$$\left|z_0^M\right| = \left|z_0\right|^M = \left|\frac{b+s_0}{b-s_0}\right| = \left|\frac{b+\mathrm{Re}\{s_0\}+j\,\mathrm{Im}\{s_0\}}{b-\mathrm{Re}\{s_0\}-j\,\mathrm{Im}\{s_0\}}\right|$$

$$= \sqrt{\frac{\left(b+\mathrm{Re}\{s_0\}\right)^2+\left(\mathrm{Im}\{s_0\}\right)^2}{\left(b-\mathrm{Re}\{s_0\}\right)^2+\left(\mathrm{Im}\{s_0\}\right)^2}}$$

- For $M > 0$; want $\left|z_0\right| < 1 \Rightarrow \left|z_0\right|^M < 1$

$$\left(b+\mathrm{Re}\{s_0\}\right)^2+\cancel{\left(\mathrm{Im}\{s_0\}\right)^2} < \left(b-\mathrm{Re}\{s_0\}\right)^2+\cancel{\left(\mathrm{Im}\{s_0\}\right)^2}$$

$$\cancel{b^2}+\cancel{2}b\,\mathrm{Re}\{s_0\}+\cancel{\mathrm{Re}^2\{s_0\}} < \cancel{b^2}-\cancel{2}b\,\mathrm{Re}\{s_0\}+\cancel{\mathrm{Re}^2\{s_0\}}$$

$$b\,\mathrm{Re}\{s_0\} < -b\,\mathrm{Re}\{s_0\}$$

$$2b\,\mathrm{Re}\{s_0\} < 0$$

$$b > 0\ ;\mathrm{Re}\{s_0\} < 0$$

- For $M < 0$; want $\left|z_0\right| < 1 \Rightarrow \left|z_0\right|^M > 1$

$$\left(b+\mathrm{Re}\{s_0\}\right)^2+\cancel{\left(\mathrm{Im}\{s_0\}\right)^2} > \left(b-\mathrm{Re}\{s_0\}\right)^2+\cancel{\left(\mathrm{Im}\{s_0\}\right)^2}$$

$$\cancel{b^2}+\cancel{2}b\,\mathrm{Re}\{s_0\}+\cancel{\mathrm{Re}^2\{s_0\}} > \cancel{b^2}-\cancel{2}b\,\mathrm{Re}\{s_0\}+\cancel{\mathrm{Re}^2\{s_0\}}$$

$$b\,\mathrm{Re}\{s_0\} > -b\,\mathrm{Re}\{s_0\}$$

$$2b\,\mathrm{Re}\{s_0\} > 0$$

$$b < 0;\mathrm{Re}\{s_0\} < 0$$

Equalization

- Design $\hat{H}(w)$ to undo effect of $\hat{G}(w)$
- Can set $H(z) = \dfrac{1}{G(z)}$
  - Ex. work if $G(z) = \dfrac{\mathrm{z}^{\mathrm{a}\geq \mathrm{d}}}{\text{polynomial (z) degree} = \mathrm{d}}$

    $H(z) = k_0 + k_1 z^{-1} + \ldots$ and $h[n] = k_0\delta[n] + k_1\delta[n-1] + \ldots$
  - Not always get causal/stable answer
    - Ex. $G(z) = \dfrac{\mathrm{z}^{\mathrm{a}<\mathrm{d}}}{\text{polynomial (z) degree} = \mathrm{d}}$

      Get z^+ in $H(z)$ and $h[n]$ is not causal

      solution: design so that $H(z)G(z) = z^{a-d}$ and then the result is simply a delay

Phase

- In general, we have $\hat{H}(w) = \left|\hat{H}(w)\right| e^{-jf(w)}$
  - $f(w)$ is 2π-periodic,

    not uniquely determined due to 2π-multiple ambiguity
- Causal real-valued $h[n]$ cannot have zero phase $\hat{H}(w) = \left|\hat{H}(w)\right|$ nor constant phase $\hat{H}(w) = \left|\hat{H}(w)\right| e^{-jb}$ unless $h[n] = K_0 d[n]$
  - $h[n]$ real $\Rightarrow$ $\hat{H}(-w) = \hat{H}^*(w)$
- Case when $\hat{H}(w) = \left|\hat{H}(w)\right| e^{-jaw}$; a $=$ $\mathbf{n_0}$, a (positive) integer
  - If $w[n] \rightarrow \boxed{\left|\hat{H}_{des}(w)\right|} \rightarrow y_{des}[n]$, then

    $w[n] \rightarrow \boxed{\hat{H}(w) = \left|\hat{H}_{des}(w)\right| e^{-jn_0 w}} \rightarrow y_{des}[n - n_0]$

    $\Rightarrow$ simple time-delay

    Proof

    $$\hat{Y}_{des}(w) = \hat{W}(w)\left|\hat{H}_{des}(w)\right|$$

    $$\hat{Y}'(w) = \hat{W}(w)\left|\hat{H}_{des}(w)\right| e^{-jn_0 w} = \hat{Y}_{des}(w) e^{-jn_0 w} \xrightarrow{DTFT^{-1}} y_{des}[n - n_0]$$

    by time-shift rule.
  - $h[n] = h_{des}[n - n_0]$

    $$h_{des}[n] \overset{DTFT}{\rightleftharpoons} \hat{H}_{des}(w) = \left|\hat{H}_{des}(w)\right|$$

    $$h[n] \xrightarrow{DTFT} \left|\hat{H}_{des}(w)\right| e^{-jn_0 w} \xrightarrow{DTFT^{-1}} h_{des}[n - n_0]$$

    - If $h_{des}[n]$ is FIR, then $h[n] = h_{des}[n - n_0]$ will be causal for n_0 large enough
  - $e^{jnw_0} \rightarrow \boxed{e^{-jn_0 w}} \rightarrow e^{-jn_0 w_0} e^{jnw_0} = e^{j(n - n_0)w_0}$
- Case when $\hat{H}(w) = \left|\hat{H}(w)\right| e^{-jaw}$; **real** a
  - $\hat{H}(w) = e^{-jaw} \xrightarrow{DTFT^{-1}} h[n] = \dfrac{1}{2p} \displaystyle\int_{-p}^{p} e^{-jaw} e^{jnw} dw = \dfrac{\sin(p(n - a))}{p(n - a)}$
    - for integer $a = n_0$, $h[n] = d[n - n_0]$
  - $w[n] \rightarrow \underset{\underset{T}{\uparrow}}{\boxed{D/C}} \xrightarrow{w_c(t)} \boxed{e^{-jaT\Omega}} \xrightarrow{y_c(t) = w_c(t - aT)} \underset{\underset{T}{\uparrow}}{\boxed{C/D}} \rightarrow y[n] = y_c(nT)$

    Proof

$$\hat{W}_c(\Omega) = \hat{W}_R(\Omega) = \begin{cases} T\hat{W}(\Omega T) & -\frac{p}{T} \le \Omega \le \frac{p}{T} \\ 0 & |\Omega| > \frac{p}{T} \end{cases}$$

$$\hat{Y}_c(\Omega) = \hat{W}_c(\Omega) e^{-jaT\Omega} = \begin{cases} Te^{-ja\Omega T}\hat{W}(\Omega T) & -\frac{p}{T} \le \Omega \le \frac{p}{T} \\ 0 & |\Omega| > \frac{p}{T} \end{cases}$$

$$\hat{Y}(w) = \sum_{k=-\infty}^{\infty} \frac{1}{T} \hat{Y}_c\left(\frac{w}{T_d} + k\frac{2p}{T}\right) = e^{-jaw}\hat{W}(w) \; ; |w| \le p$$

- $e^{jnw_0} \to \boxed{e^{-jaw}} \to \left(e^{-jaw_0}\right) e^{jnw_0} = e^{j(n-a)w_0}$
- $e^{jnw_0} \to \underset{\uparrow T}{\boxed{D/C}} \xrightarrow{e^{j\frac{t}{T}w_0}} \boxed{e^{-jaT\Omega}} \xrightarrow{e^{j\frac{t-aT}{T}w_0}} \underset{\uparrow T}{\boxed{C/D}} \to e^{j\frac{nT-aT}{T}w_0} = e^{j(n-a)w_0}$

- Generalized linear phase: $\hat{H}(w) = A(w)e^{-j(aw+b)}$
  - Real a, b, $A(w)$
  - Can be expressed with $b = 0$ or $\frac{p}{2}$ for real $h[n]$

    Proof

    $h[n]$ is real $\Rightarrow \hat{H}(-w) = \hat{H}^*(w)$

    $A(-w)e^{-j(-aw+b)} = A(w)e^{j(aw+b)}$

    $A(-w) = A(w)e^{j2b} \quad real$

    So e^{j2b} is real = trivial 0, 1, or –1

    - If $e^{j2b} = 1$, $e^{jb} = \pm 1 \Rightarrow$ absorb e^{jb} into $A(w)$ and $b = 0$
    - If $e^{j2b} = -1$, $e^{jb} = \pm j \Rightarrow$ absorb sign into $A(w)$ and $e^{jb} = j \Rightarrow b = \frac{p}{2}$
- Truly linear phase when $b = 0$ and $A(w) \ge 0 \ \forall w$
  - Let $A(w) = |\hat{H}(w)|$, then $\hat{H}(w) = |\hat{H}(w)|e^{-jaw}$

FIR filter design

FIR filter with generalized linear phase (g.l.p.)

- Every generalized-linear-phase FIR filter $\hat{H}(w) = A(w)e^{-j(aw+b)}$ is of one of these 4 types:

Type	I	II	III	IV
$h[M+m]=$ M = midpoint	$h_I[M-m]$	$h_{II}[M+1-m]$	$-h_{III}[M-m]$	$-h_{IV}[M+1-m]$
h[n] duration = N	odd	even	odd	even
Midpoint @	M	M, M+1	M	M, M+1
h[n] around midpoint	even	even	odd	odd
#unknown h[n] = η	$\frac{N+1}{2}$	$\frac{N}{2}$	$\frac{N-1}{2}$	$\frac{N}{2}$
A(ω) about 0	even	even	odd	odd
A(ω) @ 0	≠0	≠0	0	0
A(ω) about π	even	odd	odd	even
A(ω) @ π	≠0	0	0	≠0
α	M	M+$\frac{1}{2}$	M	M+$\frac{1}{2}$
β	0	0	$\frac{p}{2}$	$\frac{p}{2}$
Think about A(ω) as	cos(ω)	$\cos\left(\frac{w}{2}\right)$	sin(ω)	$\sin\left(\frac{w}{2}\right)$
filter	H,L	L		H
After shifted by π in ω or ×(-1)^n in n	I	IV	I	II

- $A_I(w)=h[M]+\sum_{m>0}\{2h[M+m]\cos(mw)\}$
- $A_{II}(w)=\sum_{m>0}\left\{2h[M+m]\cos\left(\left(m-\frac{1}{2}\right)w\right)\right\}$
- $A_{III}(w)=\sum_{m>0}\{2h[M+m]\sin(mw)\}$
- $A_{IV}(w)=\sum_{m>0}\left\{2h[M+m]\sin\left(\left(m-\frac{1}{2}\right)w\right)\right\}$

- Type I
  - $h_I[n]$
    - Odd duration N
    - Even-symmetric about midpoint $h[M]$: $h_I[M+m]=h_I[M-m]$, $\forall m>0$
  - $a = M, b = 0$

- $A_I(w) = h[M] + \sum_{m>0} \{2h[M+m]\cos(mw)\}$

Proof

$$\hat{H}_I(w) = \sum_n h[n] e^{-jnw}$$

$$= h[M]e^{-jMw} + \sum_{m>0}\left\{h[M+m]e^{-j(M+m)w} + h[M-m]e^{-j(M-m)w}\right\}$$

$$= h[M]e^{-jMw} + e^{-jMw}\sum_{m>0}\left\{h[M+m]e^{-jmw} + h[M+m]e^{+jmw}\right\}$$

$$= \left[h[M] + \sum_{m>0}\{2h[M+m]\cos(mw)\}\right]e^{-jMw}$$

- Even about $w = 0$: $A(0+w) = A(0-w)$

  Proof $\cos(m(-w)) = \cos(mw)$

- Even about $w = \pi$: $A(p+w) = A(p-w)$

  Proof

$$\cos(m(p+w)) = \frac{1}{2}\left(e^{j(mp+mw)} + e^{-j(mp+mw)}\right)$$

$$= \frac{1}{2}\left(e^{jmp}e^{jmw} + e^{-jmp}e^{-jmw}\right) = (-1)^m \cos(mw)$$

$$\cos(m(p-w)) = \frac{1}{2}\left(e^{j(mp-mw)} + e^{-j(mp-mw)}\right)$$

$$= \frac{1}{2}\left(e^{jmp}e^{-jmw} + e^{-jmp}e^{+jmw}\right) = (-1)^m \cos(mw)$$

- Periodic with period 2π: $A(w+2p) = A(w)$

  Proof $\cos(m(w+2p)) = \cos(mw + m2p) = \cos(mw)$

- Type II
  - $h[n]$
    - Even duration N
    - Even-symmetric about midpoint $h[M] = h[M+1]$:

      $h_{II}[M+m] = h_{II}[M+1-m], \forall m > 0$

- $a = M + \frac{1}{2}, \ b = 0$

- $A_{II}(w) = \sum_{m>0}\left\{2h[M+m]\cos\left(\left(m-\frac{1}{2}\right)w\right)\right\}$

Proof

$$\hat{H}_{II}(w) = \sum_{n} h[n] e^{-jnw}$$

$$= \sum_{m>0} \left\{ h[M+m] e^{-j(M+m)w} + h[M+1-m] e^{-j(M+1-m)w} \right\}$$

$$= e^{-j\left(M+\frac{1}{2}\right)w} \sum_{m>0} \left\{ h[M+m] e^{-j\left(m-\frac{1}{2}\right)w} + h[M+m] e^{+j\left(m-\frac{1}{2}\right)w} \right\}$$

$$= \left[\sum_{m>0} \left\{ 2h[M+m] \cos\left(\left(m-\frac{1}{2} \right) w \right) \right\} \right] e^{-j\left(M+\frac{1}{2}\right)w}$$

- Even about w = 0: $A(0+w) = A(0-w)$

  Proof $\cos\left(\left(m-\frac{1}{2} \right)(-w) \right) = \cos\left(\left(m-\frac{1}{2} \right) w \right)$

- Odd about w = π: $A(p+w) = -A(p-w)$

  Proof

$$\cos\left(\left(m-\frac{1}{2} \right)(p+w) \right) = \cos\left(mp - \frac{p}{2} + mw - \frac{w}{2} \right)$$

$$= \mathrm{Re}\left\{ e^{jmp} e^{-j\frac{p}{2}} e^{jmw} e^{-j\frac{w}{2}} \right\}$$

$$= \mathrm{Re}\left\{ (-1)^m (-j) e^{jmw} e^{-j\frac{w}{2}} \right\}$$

$$= (-1)^m \sin\left(mw - \frac{w}{2} \right)$$

$$\cos\left(\left(m-\frac{1}{2} \right)(p-w) \right) = \cos\left(mp - \frac{p}{2} - mw + \frac{w}{2} \right)$$

$$= \mathrm{Re}\left\{ e^{jmp} e^{-j\frac{p}{2}} e^{-jmw} e^{+j\frac{w}{2}} \right\}$$

$$= \mathrm{Re}\left\{ (-1)^m (-j) e^{-jmw} e^{+j\frac{w}{2}} \right\}$$

$$= (-1)^m \sin\left(-mw + \frac{w}{2} \right)$$

$$= -(-1)^m \sin\left(mw - \frac{w}{2} \right)$$

- Periodic with period 4π: $A(w+4p) = A(w)$

Proof $\cos\left(\left(m-\frac{1}{2}\right)(w+4\pi)\right)=\cos\left(\left(m-\frac{1}{2}\right)w+4m\pi-2\pi\right)$

$$=\cos\left(\left(m-\frac{1}{2}\right)w\right)$$

- Type III
  - $h[n]$
    - Odd duration N
    - $h[M] = 0$
    - Odd-symmetric about midpoint $h[M] = 0$: $h_{III}[M+m]=-h_{III}[M-m]$, $\forall m > 0$
  - $a = M, b = \frac{\pi}{2}$
  - $A_{III}(w)=\sum_{m>0}\left\{2h[M+m]\sin(mw)\right\}$

Proof

$$\hat{H}_{III}(w)=\sum_{n}h[n]e^{-jnw}$$

$$=h[M]e^{-jMw}+\sum_{m>0}\left\{h[M+m]e^{-j(M+m)w}+h[M-m]e^{-j(M-m)w}\right\}$$

$$=e^{-jMw}\sum_{m>0}\left\{h[M+m]e^{-jmw}-h[M+m]e^{+jmw}\right\}$$

$$=\left[\sum_{m>0}\left\{2(-j)h[M+m]\sin(mw)\right\}\right]e^{-jMw}$$

$$=\left[\sum_{m>0}\left\{2h[M+m]\sin(mw)\right\}\right]e^{-j\left(Mw+\frac{\pi}{2}\right)}$$

- Odd about $w = 0$: $A(0+w)=-A(0-w)$
- Odd about $w = \pi$: $A(\pi+w)=-A(\pi-w)$
- Periodic with period 2π: $A(w+2\pi)=A(w)$

- Type IV
  - $h[n]$
    - Even duration N
    - Odd-symmetric about midpoint $h[m]=-h[m+1]$:
      $h_{IV}[M+m]=-h_{IV}[M+1-m]$, $\forall m > 0$

- $a = M + \frac{1}{2}$, $b = \frac{p}{2}$
- $A_{IV}(w) = \sum_{m>0} \left\{ 2h[M+m]\sin\left(\left(m - \frac{1}{2}\right)w\right)\right\}$

Proof

$$\hat{H}_{IV}(w) = \sum_n h[n]e^{-jnw}$$
$$= \sum_{m>0} \left\{ h[M+m]e^{-j(M+m)w} + h[M+1-m]e^{-j(M+1-m)w} \right\}$$
$$= e^{-j\left(M+\frac{1}{2}\right)w} \sum_{m>0} \left\{ h[M+m]e^{-j\left(m-\frac{1}{2}\right)w} - h[M+m]e^{+j\left(m-\frac{1}{2}\right)w} \right\}$$
$$= \left[\sum_{m>0} \left\{ 2(-j)h[M+m]\sin\left(\left(m-\frac{1}{2}\right)w\right)\right\}\right] e^{-j\left(M+\frac{1}{2}\right)w}$$
$$= \left[\sum_{m>0} \left\{ 2h[M+m]\sin\left(\left(m-\frac{1}{2}\right)w\right)\right\}\right] e^{-j\left(\left(M+\frac{1}{2}\right)w+\frac{p}{2}\right)}$$

- Odd about $w = 0$: $A(0+w) = -A(0-w)$
- Even about $w = \pi$: $A(p+w) = A(p-w)$
- Periodic with period 4π: $A(w+4p) = A(w)$

- a = mid-location of the duration interval
- $b = \frac{p}{2}$ when having odd-symmetric h[n] about midpoint(s)

  To see this, odd ⇒ negative sign in the middle ⇒ sin ⇒ -j
- A(0) = 0 (III and IV) ⇒ block DC ⇒ bad low-pass filter
- A(π) = 0 (II and III) ⇒ bad high-pass filter
- Type I isn't the best since pass w = 0, π
- Cascading g.l.p FIR filters acts as a g.l.p FIR

  $$\hat{H}(w) = \prod_i \hat{H}_i(w) = \left(\prod_i A_i(w)\right) e^{-j\left(\left(\sum_i a_i\right)w + \sum_i b_i\right)}$$
- Adding ("+") g.l.p FIR filters may not acts as a g.l.p FIR

  To see this, consider the symmetry of resulted $h[n]$

Filter Design technique

- Given $\hat{H}_{des}(w)$

- Targeting FIR g.l.p. filters

Frequency-sampling Design

- Find real $h[n]$ FIR g.l.p. filter with duration interval $0 \le n < N$

  and $\left|\hat{H}\left(k\frac{2p}{N}\right)\right| = \left|\hat{H}_{des}\left(k\frac{2p}{N}\right)\right|$ $0 \le k < N$

  (always exist)

- Step

  1) Given N
  2) Pick filter type according to the <u>magnitude</u> of $\hat{H}_{des}(w)$ around 0 , π

     $\Rightarrow$ know α, β
  3) Set real $\tilde{A}(w)$ which
     - has required symmetry for the targeted filter type
     - $\tilde{A}(w) = \left|\tilde{A}(w)\right| = \left|\hat{H}_{des}(w)\right|$
  4) Get N equations from: $A\left(k\frac{2p}{N}\right) = \tilde{A}\left(k\frac{2p}{N}\right)$

     Note that we now look at $\tilde{A}(w)$ for $0 < \omega < 2\pi$
  5) Do one of the following:

     5.1) Solve for h[n]-values from the above N equations, using linear algebra.

     5.2) Set $\hat{G}[k] = \tilde{A}\left(k\frac{2p}{N}\right)e^{-j\left(a k\frac{2p}{N}+b\right)}$; $0 \le k < N$

     $$h[n] = \frac{1}{N}\sum_{k=0}^{N-1}\hat{G}[k]y_N^{+nk} ;\ 0 \le n < N$$

Time-domain least-squares design

- Minimize $\sum_{\ell=0}^{L-1}\left\|\hat{H}(w_\ell)\right| - \left|\hat{H}_{des}(w_\ell)\right\|$ for general set of w-points: ω_ℓ , $0 \le \ell < L$; L can > N
  - Match (approximately) at a general set of w-points: w_ℓ , $0 \le \ell < L$ not necessarily uniformly spaced $\Rightarrow$ can prioritize matching regions
- Find $h[n]$, $0 \le n < N$, such that $\sum_{\ell=0}^{L-1}\left|A(w_\ell) - \tilde{A}(w_\ell)\right|$ is minimized
- Let $\alpha_\ell = \tilde{A}(w_\ell)$, $\underline{h} = \begin{pmatrix} h[first] \\ \vdots \\ h[first+h-1] \end{pmatrix}$, $\underline{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_k \end{pmatrix}$,

$\Gamma_{L\times h}$ = matrix with entries from coefficients on h's

- $\Rightarrow$ Minimize $\|\Gamma \underline{h} - \underline{a}\|^2 \rightarrow \hat{\underline{h}} = \left(\Gamma^T \Gamma\right)^{-1} \Gamma^T \underline{a}$

Weighted time domain least squares filter design

- Given weights $\varepsilon_\ell > 0$. LHW $\begin{pmatrix} e_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & e_L \end{pmatrix}$
- Choose $\underline{h}$ to minimize $\sum_{\ell=0}^{L-1} e_\ell \left|[\Gamma \underline{h}]_\ell - a_\ell\right|^2$ or $(\Gamma \underline{h} - \underline{a})^T \mathrm{E} (\Gamma \underline{h} - \underline{a}) \Rightarrow$

  $\hat{\underline{h}} = \left(\Gamma^T \mathrm{E} \Gamma\right)^{-1} \Gamma^T \mathrm{E} \underline{a}$

Windowing

- Minimize $\frac{1}{2p} \int_{-p}^{p} \left|\hat{H}(w) - \hat{H}_{des}(w)\right|^2 dw$ by

  Windowing $h_{des}[n]$ with **rectangular** windowing function $\square_L[n] = \begin{cases} 1 & |n| < L \\ 0 & |n| \geq L \end{cases} \Rightarrow$

  $h[n] = \square_L[n] h_{des}[n] = \begin{cases} h_{des}[n] & |n| < L \\ 0 & |n| \geq L \end{cases}$ = truncated version of $h_{des}[n]$

  Proof Use Parseval Identity:

$$\frac{1}{2p} \int_{-p}^{p} \left|\hat{H}(w) - \hat{H}_{des}(w)\right|^2 dw = \sum_{n=-\infty}^{\infty} \left|h[n] - h_{des}[n]\right|^2$$

$$= \sum_{n=-(L-1)}^{L-1} \left|h[n] - h_{des}[n]\right|^2 + \sum_{|n| \geq L} \left|0 - h_{des}[n]\right|^2$$

  To minimize, set $h[n] = h_{des}[n]$; $-L < n < L$

- $h[n]$ isn't causal $\Rightarrow$ shift it by (L-1) $\rightarrow 0 \leq n < 2L-1$
- $\hat{H}(w) = \sum_{n=-(L-1)}^{(L-1)} h_{des}[n] e^{-jnw}$ = a partial sum of $\hat{H}_{des}(w)$, which is $\sum_{n=-\infty}^{\infty} h_{des}[n] e^{-jnw}$
- $\hat{\square}(w) = \dfrac{\sin\left(\left(L - \frac{1}{2}\right) w\right)}{\sin\left(\frac{w}{2}\right)} = \dfrac{\sin\left(\frac{N}{2} w\right)}{\sin\left(\frac{w}{2}\right)}$

  Proof

$$\sum_{k=0}^{(2L-1)-1} d[n-k] \xrightarrow{DTFT} e^{-j\frac{2L-1-1}{2}w}\left(\frac{\sin\left(\frac{2L-1}{2}w\right)}{\sin\left(\frac{1}{2}w\right)}\right)$$

$$\sum_{k=-(L-1)}^{L-1} d[n-k] \xrightarrow{DTFT} \cancel{e^{-j(L-1)w}}\left(\frac{\sin\left(\frac{2L-1}{2}w\right)}{\sin\left(\frac{1}{2}w\right)}\right)\cancel{e^{j(L-1)w}}$$; time-shift rule

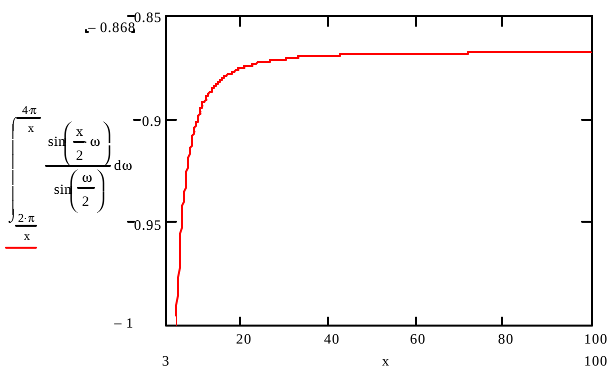
- $\hat{\square}(w) = \dfrac{\sin\left(\left(L-\frac{1}{2}\right)w\right)}{\sin\left(\frac{w}{2}\right)} = 0$ iff $\left(L-\frac{1}{2}\right)w = mp,\ m \neq 0 \Rightarrow$

  $w = \dfrac{mp}{L-\frac{1}{2}} = \dfrac{2mp}{N}$, $m \neq 0$

- Central lobe's width = $\dfrac{2p}{L-\frac{1}{2}} = \dfrac{4p}{2L-1} = \dfrac{4p}{N} \Rightarrow$ decrease as N increase

- Area under one side of first side-lobe = $\displaystyle\int_{\frac{p}{L-\frac{1}{2}}}^{\frac{2p}{L-\frac{1}{2}}} \frac{\sin\left(\left(L-\frac{1}{2}\right)w\right)}{\sin\left(\frac{w}{2}\right)}dw = \int_{\frac{2p}{N}}^{\frac{4p}{N}} \frac{\sin\left(\frac{N}{2}w\right)}{\sin\left(\frac{w}{2}\right)}dw$

  $\Rightarrow$ roughly the same @ -0.868 as N increases



- $h[n] = \square_L[n]h_{des}[n] \xrightarrow{DTFT} \hat{H}(w) = \dfrac{1}{2p}\displaystyle\int_{-p}^{p} \hat{H}_{des}(m)\hat{\square}(w-m)\,dm$

- $\hat{H}(w) = \dfrac{1}{2p}\displaystyle\int_{-p}^{p} \hat{H}_{des}(m)\hat{\square}(w-m)\,dm$

  Start with w = 0 and increasing it.

Assume that the side lobes beyond the first one don't contribute too much.

- First, central and two first-side-lobe of $\hat{□}(w - m)$ is in the passband of $\hat{H}_{des}(m)$, so LQWHJUDO $\approx \begin{pmatrix}\text{area under}\\ \text{main lobe}\end{pmatrix} - 2\begin{pmatrix}\text{area under}\\ \text{first-side-lobe}\end{pmatrix}$
- Next, the right first-side-lobe is going out of $\hat{H}_{des}(m)$'s passband so the integral increase until all of the right first-side-lobe is gone out of $\hat{H}_{des}(m)$'s passband. Then, integral $\approx \begin{pmatrix}\text{area under}\\ \text{main lobe}\end{pmatrix} - \begin{pmatrix}\text{area under}\\ (\text{left})\,\text{first-side-lobe}\end{pmatrix}$

  so, height of the overshoot is proportional to the area of the first-side lobe
- Then, Central lobe start going out of $\hat{H}_{des}(m)$'s passband so the integral start to decrease again (big decrease). This is where the transition region occurs and it is proportional to the width of the main lobe.
- Finally, all of the main lobe is gone out of $\hat{H}_{des}(m)$'s passband. The left first-side lob starts to get out, so the integral increases again because less negative part is included.

- Gibbs Phenomenon (9% overshoot) at jump
- Bigger N ⇒ narrower central lobe ⇒ narrower transition region
- Same first-side-lobe area ⇒ same peak overshoot (Gibbs); ∀N

	□$_L$[n] for –L < n < L (= 0 for |n| ≥ L)	P DLQ-OREH Z LGWK	WVLG -OREH KHLJKWVLGH - OREHDUHD
5 HFWDQJXODU		$\frac{4p}{N}$	- G
7 UDQJXODU %DUWOHWW	$-\frac{\|n\|}{L}$	$\frac{8p}{N}$	- G
+ DQQ + DQQLQJ	$\frac{1}{2}+\frac{1}{2}\cos\frac{np}{2}$	$\frac{8p}{N}$	- G
Hamming	$.54+.46\cos\frac{np}{2}$	$\frac{8p}{N}$	- G

0 LQLP D[ ) LOWHUGHVLJQ

- w_c = cutoff

  w_p = last $w < w_c$ (in passband) where $\hat{H}(w) = 1$

  w_s = first $w > w_c$ (in stopband) where $\hat{H}(w) = 0$

  w_s-w_p ~ transition region width

  d_1 = passband ripple max

  d WRSEDQGULSSOHP D[

- 3URWRW\SHSUREOHP
  - DHVLJQDGXUDWLRQ-1),5 ILOWHU
  - transition region width w_s-w_p < a specific amount
  - minimizes the maximum of d_1 and d_2
- 3URSHUWLHVRIVROXWLRQ
  - Equiripple in both passband and stopband
  - d d
  - number of ripples between 0 and w_c is N
  - ELJJHU N ⇒ PDOOHU d d UHLSSOHV

Signal flow graph

- Any signal flow graph describing filter is a realization of the filter
- Minimal/canonical realization ⇒ fewest possible delay branch "$\xrightarrow{z^{-1}}$"
- #delay branches ≈ amount of memory required via the given signal flow graph
- In general, $H(z) = \dfrac{p(z)}{q(z)} = \underbrace{p(z)}_{FIR}\underbrace{\left(\dfrac{1}{q(z)}\right)}_{IIR}$, a proper rational function
- Order of the filter = degree of $q(z) = N$ (assume fraction is in lowest terms)
  - Ex.
- Filter is FIR ⇔ $q(z) = z^n$
- Minimal realization have # delay branches = Order of the filter = N
- $y[n] + a_1 y[n-1] + a_2 y[n-2] = b_0 w[n] + b_1 w[n-1] + b_2 w[n-2]$

  $H(z) = \dfrac{b_0 z^2 + b_1 z + b_2}{z^2 + a_1 z + a_2}$ ⇒ N=2

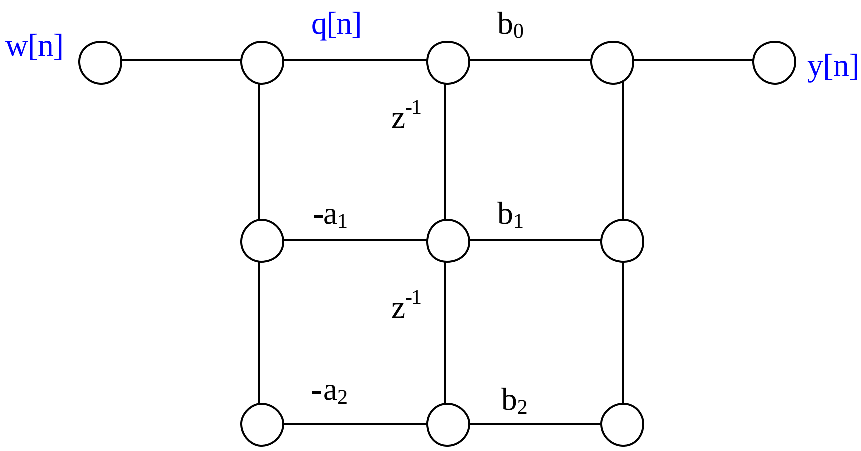
Direct Form II: Controllable canonical realization

- $Y(z) = \dfrac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}} W(z) = \left(b_0 + b_1 z^{-1} + b_2 z^{-2}\right)\underbrace{\left(\dfrac{W(z)}{1 + a_1 z^{-1} + a_2 z^{-2}}\right)}_{Q(z)}$
  - $Q(z) = \dfrac{W(z)}{1 + a_1 z^{-1} + a_2 z^{-2}}$

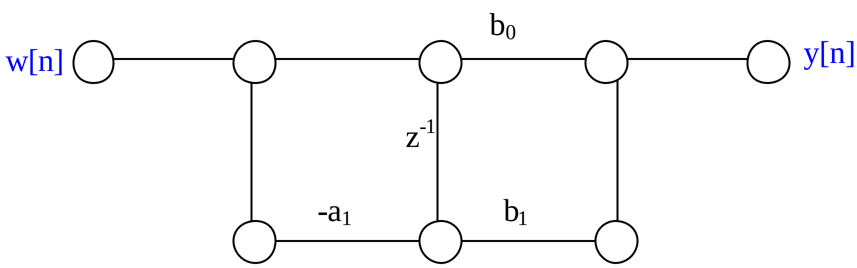
    $Q(z) + a_1 z^{-1} Q(z) + a_2 z^{-2} Q(z) = W(z)$

    $q[n] + a_1 q[n-1] + a_2 q[n-2] = w[n]$
  - $q[n] = w[n] - a_1 q[n-1] - a_2 q[n-2]$

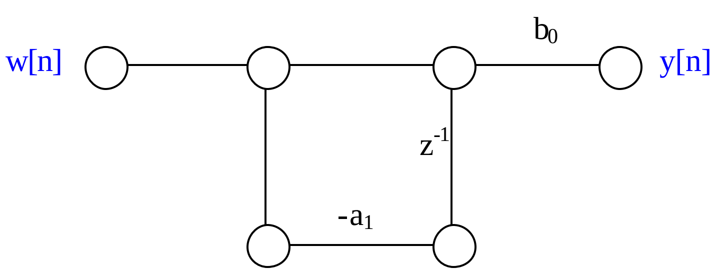
$$Y(z) = \left(b_0 + b_1 z^{-1} + b_2 z^{-2}\right) Q(z)$$
$$Y(z) = b_0 Q(z) + b_1 z^{-1} Q(z) + b_2 z^{-2} Q(z)$$
$$y[n] = b_0 q[n] + b_1 q[n-1] + b_2 q[n-2]$$



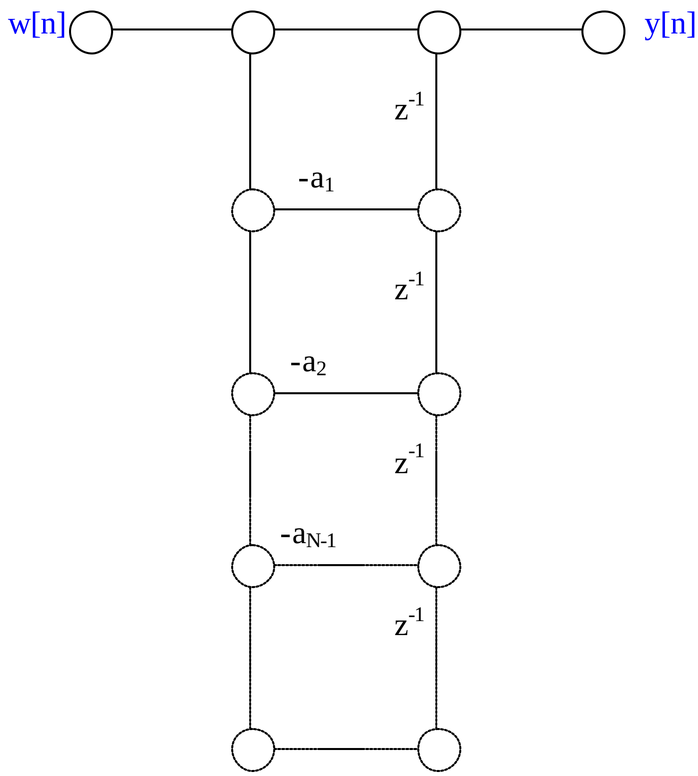
- $H(z) = \dfrac{b_0 z + b_1}{z + a_1} = \dfrac{b_0 + b_1 z^{-1}}{1 + a_1 z^{-1}}$



- $H(z) = \dfrac{b_0 z}{z + a_1} = \dfrac{b_0}{1 + a_1 z^{-1}}$

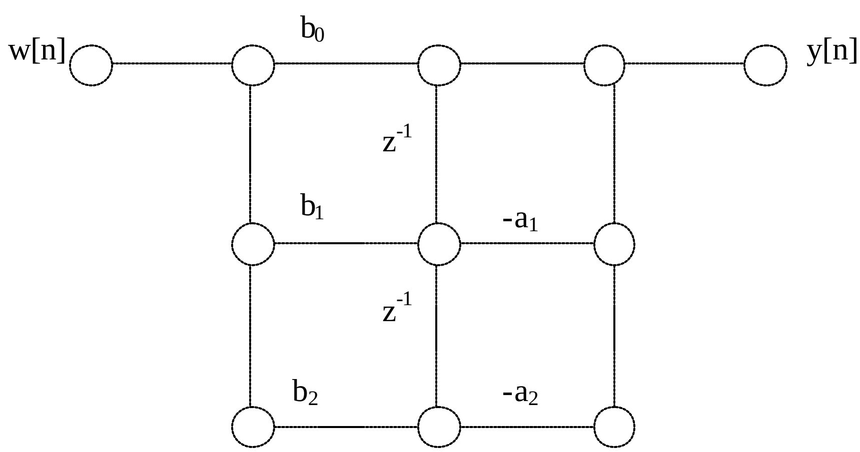


- $H(z) = \dfrac{1}{1 + a_1 z^{-1} + \ldots + a_N z^{-N}}$



Transposed Direct Form II: Observable canonical realization

- $y[n] = b_0 w[n] + \left(b_1 w[n-1] - a_1 y[n-1]\right) + \left(b_2 w[n-2] - a_2 y[n-2]\right)$

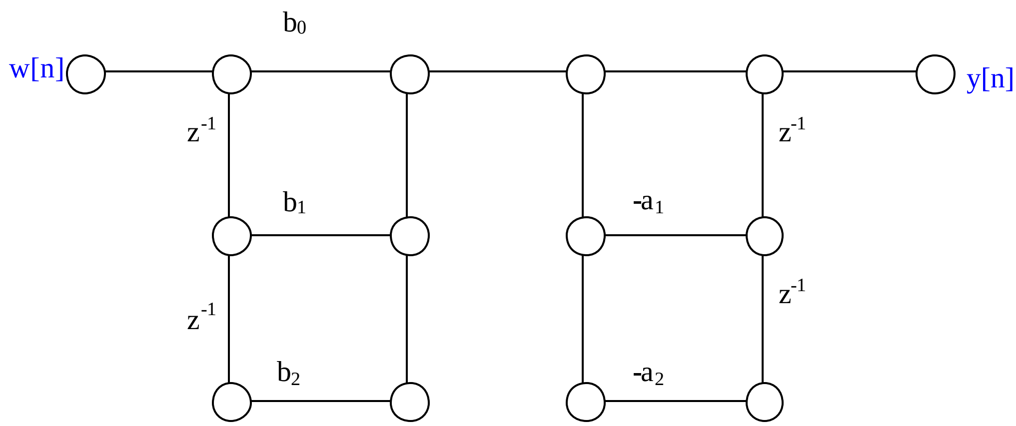


- Guarantees that it also realizes $H(z)$
- To get from direct form II, just reverse all the arrows and switch roles of $w[n]$ and $y[n]$

' LUHFW)RUP ,

- QRQ-P LQLP DO UHDOL]DWLRQ
- $g[n] = b_0 w[n] + b_1 w[n-1] + b_2 w[n-2] = y[n] + a_1 y[n-1] + a_2 y[n-2]$

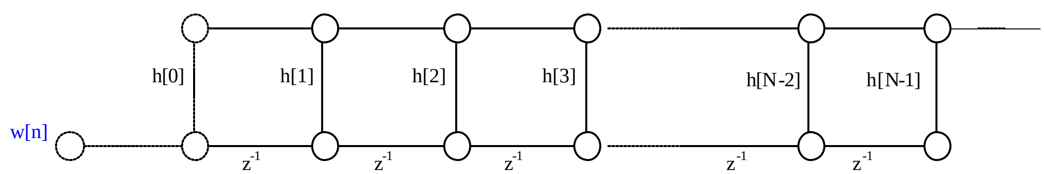
$$y[n] = g[n] - a_1 y[n-1] - a_2 y[n-2]$$



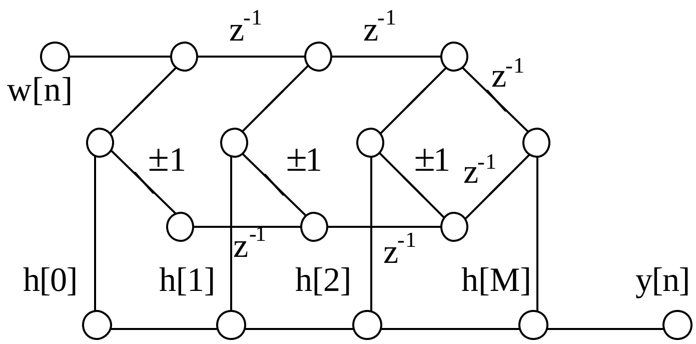
- **Cascade Realization** : a chain of direct form realizations of the individual 1st- and 2nd- order factors
- **3DUDOOHOIRUP**
  - ([SDQG $\frac{H(z)}{z}$ XVJ SDUWLDOIUDFWLRQV
  - 5HDOL]H HDFK SIHFH GLUHFWO\
  - HRRNXS LQ SDUHOOHO
- **Cascade of an FIR system with an IIR system**

$$H(z) = \frac{p(z)}{q(z)} = \underbrace{p(z)}_{FIR}\underbrace{\left(\frac{1}{q(z)}\right)}_{IIR}$$

- (Causal) **FIR Filters**
  - h[n] = $\sum_{k=0}^{N-1} h[k]d[n-k]$; H(z) = $\sum_{k=0}^{N-1} h[k]z^{-k}$; y[n] = $\sum_{k=0}^{N-1} h[n]w[n-k]$

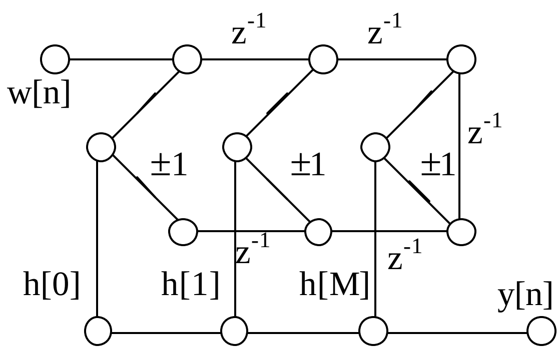


- FIR filters with generalized linear phase
  -) RU1 RGG , ,,,



use –1 for odd symmetric *h*[*n*] (III)

-) RU1 HMHQ ,, ,9



use –1 for odd symmetric *h*[*n*] (IV)