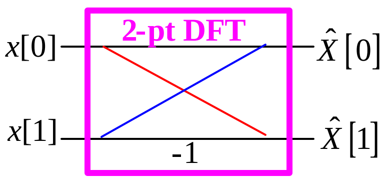
FFTs (Fast Fourier Transforms)

- 1-pt DFT: $\hat{X}[0] = x[0]$
- 2-pt DFT block

$$\hat{X}[0] = \sum_{n=0}^{1} x[n]\cancel{y_2^{-n0}} = x[0] + x[1]$$

$$\hat{X}[1] = \sum_{n=0}^{1} x[n]y_2^{-n\cancel{k}} = x[0]y_2^{-0} + x[1]y_2^{-1} = x[0] - x[1]$$



Decimation in time FFTs

- Given an N-pt signal $x[n]$, $0 \le n <$ N; assume $N = 2^L$

Let $\begin{Bmatrix} f[m] = x[2m] \\ g[m] = x[2m+1] \end{Bmatrix}$ $\quad 0 \le m < M = \dfrac{N}{2}$; each M-pt signal

$$\hat{X}[k] = \begin{cases} \hat{F}[k] + y_N^{-k}\hat{G}[k] & 0 \le k < \dfrac{N}{2} \\ \hat{F}\left[k - \dfrac{N}{2}\right] + y_N^{-k}\hat{G}\left[k - \dfrac{N}{2}\right] & \dfrac{N}{2} \le k < N \end{cases}$$

$$= \hat{F}\left[\langle k \rangle_{\frac{N}{2}}\right] + y_N^{-k}\hat{G}\left[\langle k \rangle_{\frac{N}{2}}\right] \qquad ;0 \le k < \frac{N}{2}$$

Proof

$$\hat{X}[k] = \sum_{\substack{n=0 \\ n\,even}}^{N-1} x[n]\Psi_N^{-nk} + \sum_{\substack{n=0 \\ n\,odd}}^{N-1} x[n]\Psi_N^{-nk}$$

$$= \sum_{m=0}^{\frac{N}{2}-1} x[2m]\Psi_N^{-(2m)k} + \sum_{m=0}^{\frac{N}{2}-1} x[2m+1]\Psi_N^{-(2m+1)k}$$

$$= \sum_{m=0}^{\frac{N}{2}-1} f[m]\Psi_N^{-2mk} + \Psi_N^{-k}\sum_{m=0}^{\frac{N}{2}-1} g[m]\Psi_N^{-2mk}$$

$$N \bmod 2 = 0 \;\Rightarrow\; \Psi_N^{-2mk} = \Psi_{\frac{N}{2}}^{-mk}$$

$$\hat{X}[k] = \sum_{m=0}^{\frac{N}{2}-1} f[m]\Psi_{\frac{N}{2}}^{-mk} + \Psi_N^{-k}\sum_{m=0}^{\frac{N}{2}-1} g[m]\Psi_{\frac{N}{2}}^{-mk}$$
$$= \sum_{m=0}^{\frac{N}{2}-1} f[m]\Psi_M^{-mk} + \Psi_N^{-k}\sum_{m=0}^{\frac{N}{2}-1} g[m]\Psi_M^{-mk}$$

For $0 \le k \le M-1$, $\Psi_M^{-mk} = \Psi_M^{-m\langle k\rangle_M}$.

For $M \le k \le 2M-1$, $0 \le k-M \le M-1$, and $\langle k\rangle_M = k-M$.

Thus, $\Psi_M^{-m\langle k\rangle_M} = \Psi_M^{-m(k-M)} = \Psi_M^{-mk}$ also.

$$\hat{X}[k] = \sum_{m=0}^{\frac{N}{2}-1} f[m]\Psi_M^{-mk} + \Psi_N^{-k}\sum_{m=0}^{\frac{N}{2}-1} g[m]\Psi_M^{-mk}$$
$$= \sum_{m=0}^{M-1} f[m]\Psi_M^{-m\langle k\rangle_M} + \Psi_N^{-k}\sum_{m=0}^{M-1} g[m]\Psi_M^{-m\langle k\rangle_M}$$
$$= \hat{F}\left[\langle k\rangle_M\right] + \Psi_N^{-k}\hat{G}\left[\langle k\rangle_M\right]$$

- # computation (1 computation = 1 multiply & 1 add)
  - Normal DFT: $\hat{X}[k] = \sum_{n=0}^{N-1} x[n]y_N^{-nk}$; $0 \le k < N$ $\Rightarrow$ require N computations for each k $\Rightarrow$ require N^2 computation to get $\hat{X}[k]$, $\forall$k.
  - 1-level Decimation in time FFTs $\Rightarrow$ require 2(M-pt DFT) + $N_{assembly}$ = $2\left(\frac{N}{2}\right)^2 + N = \frac{N^2}{2} + N$
  - Keep this up: #computation = $(2^L)^2 \rightarrow 2^L + L\cdot 2^L \approx N\log_2 N \ll N^2$

    To see this,

$$\underbrace{2\left[\left(2^{L-1}\right)^2\right] + 2^L}_{1^{st}-level} \rightarrow$$

$$2\left[\underbrace{2\left[\left(2^{L-2}\right)^2\right] + 2^{L-1}}_{2^{nd}-level}\right] + 2^L = 2^2\left[\left(2^{L-2}\right)^2\right] + 2\cdot 2^L \rightarrow$$

$$2^2\left[\underbrace{2\left[\left(2^{L-3}\right)^2\right] + 2^{L-2}}_{3^{rd}-level}\right] + 2\cdot 2^L = 2^3\left[\left(2^{L-3}\right)^2\right] + 3\cdot 2^L \rightarrow$$

$$2^{p-1}\left[\underbrace{2\left[\left(2^{L-p}\right)^{p-1}\right]+2^{L-(p-1)}}_{p^{th}-level}\right]+(p-1)\cdot 2^{L}=2^{p}\left[\left(2^{L-p}\right)^{p-1}\right]+p\cdot 2^{L}\rightarrow$$

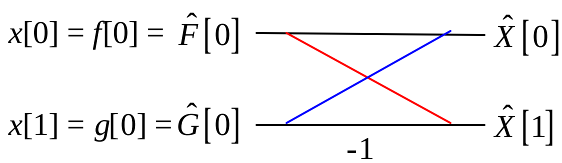
$$2^{L-1}\left[\underbrace{2\left[\left(2^{L-L}\right)^{L-1}\right]+2^{L-(L-1)}}_{L^{th}-level}\right]+(L-1)\cdot 2^{L}=2^{L}+L\cdot 2^{L}$$

For large L, this is ~ $L(2^L) = N\log_2 N << N^2$

- Butterfly Diagram:
  - N = 2

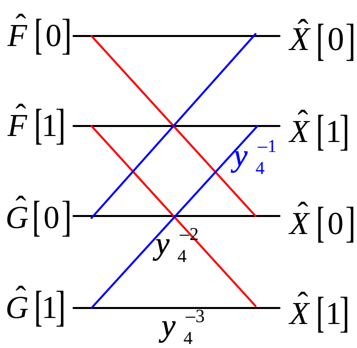
$$\hat{X}[k]=\begin{cases}\hat{F}[k]+y_{2}^{-k}\hat{G}[k] & 0\le k<1\\ \hat{F}[k-1]+y_{2}^{-k}\hat{G}[k-1] & 1\le k<N\end{cases}$$

$$\begin{bmatrix}\hat{X}[0]\\ \hat{X}[1]\end{bmatrix}=\begin{bmatrix}\hat{F}[0]+y_{2}^{-0}\hat{G}[0]\\ \hat{F}[0]+y_{2}^{-1}\hat{G}[0]\end{bmatrix}=\begin{bmatrix}\hat{F}[0]+\hat{G}[0]\\ \hat{F}[0]-\hat{G}[0]\end{bmatrix}=\begin{bmatrix}f[0]+g[0]\\ f[0]-g[0]\end{bmatrix}$$



  - N = 4

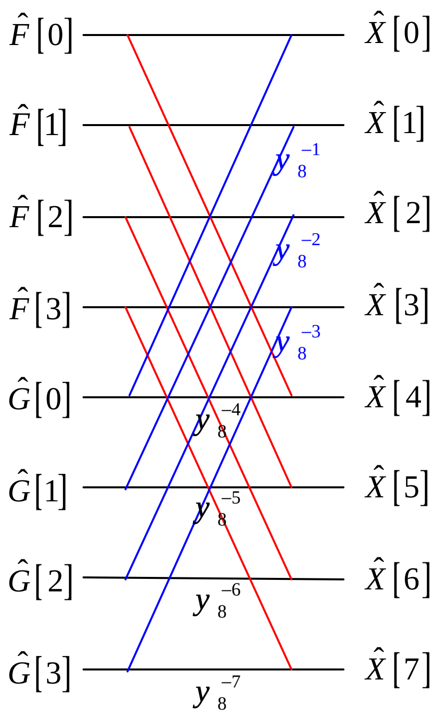
$$\begin{bmatrix}\hat{X}[0]\\ \hat{X}[1]\\ \hat{X}[2]\\ \hat{X}[3]\end{bmatrix}=\begin{bmatrix}\hat{F}[0]+y_{4}^{-0}\hat{G}[0]\\ \hat{F}[1]+y_{4}^{-1}\hat{G}[1]\\ \hat{F}[0]+y_{4}^{-2}\hat{G}[0]\\ \hat{F}[1]+y_{4}^{-3}\hat{G}[1]\end{bmatrix}=\begin{bmatrix}\hat{F}[0]+\hat{G}[0]\\ \hat{F}[1]+y_{4}^{-1}\hat{G}[1]\\ \hat{F}[0]+y_{4}^{-2}\hat{G}[0]\\ \hat{F}[1]+y_{4}^{-3}\hat{G}[1]\end{bmatrix}$$



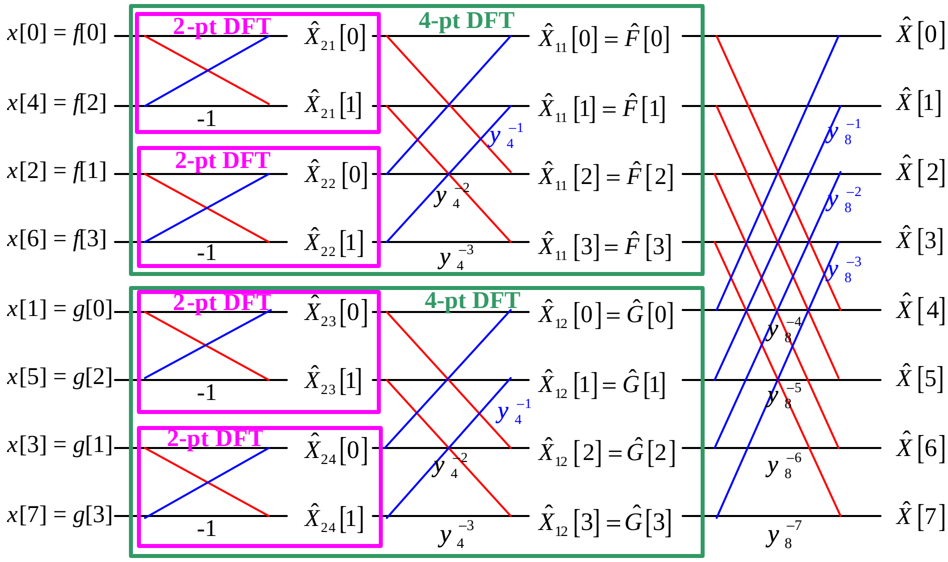
  - N = 8

$$\hat{X}[k] = \begin{cases} \hat{F}[k] + y_8^{-k}\hat{G}[k] & 0 \le k < 4 \\ \hat{F}[k-4] + y_8^{-k}\hat{G}[k-4] & 4 \le k < N \end{cases}$$

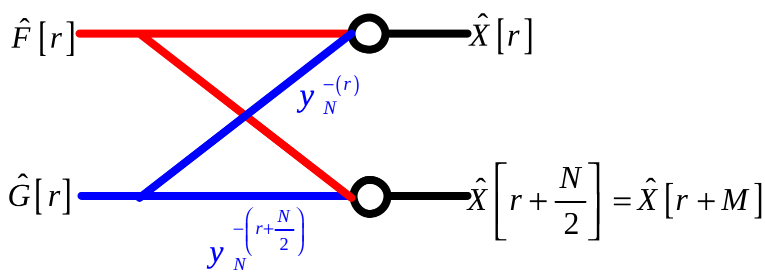
$$\begin{bmatrix} \hat{X}[0] \\ \hat{X}[1] \\ \hat{X}[2] \\ \hat{X}[3] \\ \hat{X}[4] \\ \hat{X}[5] \\ \hat{X}[6] \\ \hat{X}[7] \end{bmatrix} = \begin{bmatrix} \hat{F}[0] + y_8^{-0}\hat{G}[0] \\ \hat{F}[1] + y_8^{-1}\hat{G}[1] \\ \hat{F}[2] + y_8^{-2}\hat{G}[2] \\ \hat{F}[3] + y_8^{-3}\hat{G}[3] \\ \hat{F}[4] + y_8^{-4}\hat{G}[4] \\ \hat{F}[5] + y_8^{-5}\hat{G}[5] \\ \hat{F}[6] + y_8^{-6}\hat{G}[6] \\ \hat{F}[7] + y_8^{-7}\hat{G}[7] \end{bmatrix} = \begin{bmatrix} \hat{F}[0] + \hat{G}[0] \\ \hat{F}[1] + y_8^{-1}\hat{G}[1] \\ \hat{F}[2] + y_8^{-2}\hat{G}[2] \\ \hat{F}[3] + y_8^{-3}\hat{G}[3] \\ \hat{F}[4] + y_8^{-4}\hat{G}[4] \\ \hat{F}[5] + y_8^{-5}\hat{G}[5] \\ \hat{F}[6] + y_8^{-6}\hat{G}[6] \\ \hat{F}[7] + y_8^{-7}\hat{G}[7] \end{bmatrix}$$



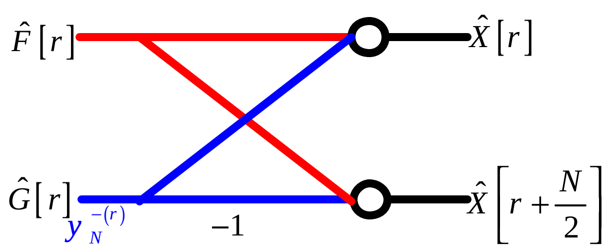
- Combination



- To get "input shuffle" of x-values, bit-reverse their indices
- Typical pair computation:



Use $y_N^{-\left(r+\frac{N}{2}\right)} = -y_N^{-r}$,



- **Power-of-3 thinking**

$f[m] = x[3m]$, $g[m] = x[3m+1]$, $h[m] = x[3m+2]$; $0 \le m < M = \dfrac{N}{3}$

$$\hat{X}[k] = \sum_{\substack{n=0 \\ \langle n\rangle_M = 0}}^{N-1} x[n]\Psi_N^{-nk} + \sum_{\substack{n=0 \\ \langle n\rangle_M = 1}}^{N-1} x[n]\Psi_N^{-nk} + \sum_{\substack{n=0 \\ \langle n\rangle_M = 2}}^{N-1} x[n]\Psi_N^{-nk}$$

$$= \sum_{m=0}^{\frac{N}{3}-1} f[m]\Psi_N^{-(3m)k} + \sum_{m=0}^{\frac{N}{3}-1} g[m]\Psi_N^{-(3m+1)k} + \sum_{m=0}^{\frac{N}{3}-1} h[m]\Psi_N^{-(3m+2)k}$$

$$= \sum_{m=0}^{M-1} f[m]\Psi_M^{-mk} + \Psi_M^{-k}\sum_{m=0}^{M-1} g[m]\Psi_M^{-mk} + \Psi_M^{-2k}\sum_{m=0}^{M-1} h[m]\Psi_M^{-mk}$$

$$= \hat{F}\left[\langle k\rangle_M\right] + \Psi_M^{-k}\hat{G}\left[\langle k\rangle_M\right] + \Psi_M^{-2k}\hat{H}\left[\langle k\rangle_M\right]$$

Decimation-in-frequency FFT

- Let $\left\{\begin{array}{l} f[m] = x[m] + x\left[m + \frac{N}{2}\right] \\ g[m] = y_N^{-m}\left(x[m] - x\left[m + \frac{N}{2}\right]\right) \end{array}\right\} \quad 0 \le m < \frac{N}{2}$

$$\hat{X}[k] = \left\{\begin{array}{ll} \hat{F}\left[\frac{k}{2}\right] & ;k \text{ even} \\ \hat{G}\left[\frac{k-1}{2}\right] & ;k \text{ odd} \end{array}\right\} \quad 0 \le k < N$$

Proof:

For k even = 2ℓ ; $0 \le \ell < \frac{N}{2}$:

$$\hat{X}[k] = \sum_{n=0}^{N-1} x[n] y_N^{-nk} = \sum_{n=0}^{N-1} x[n] y_N^{-n(2\ell)} = \sum_{n=0}^{\frac{N}{2}-1} x[n] y_N^{-n(2\ell)} + \sum_{n=\frac{N}{2}}^{N-1} x[n] y_N^{-n(2\ell)}$$

$$= \sum_{m=0}^{\frac{N}{2}-1} x[m] y_N^{-2(m\ell)} + \underbrace{\sum_{m=0}^{\frac{N}{2}-1} x[m+M] y_N^{-2(m+M)\ell}}_{m=n-\frac{N}{2}=n-M}$$

$$= \sum_{m=0}^{M-1} x[m] y_{\frac{N}{2}}^{-m\ell} + \sum_{m=0}^{M-1} x[m+M] y_{\frac{N}{2}}^{-\left(m\ell+\cancel{\frac{N}{2}\ell}\right)}$$

$$= \sum_{m=0}^{M-1} x[m] y_M^{-m\ell} + \sum_{m=0}^{M-1} x[m+M] y_M^{-m\ell}$$

$$= \sum_{m=0}^{M-1} \left(x[m] + x[m+M]\right) y_M^{-m\ell} = \sum_{m=0}^{M-1} f[m] y_M^{-m\ell}$$

$$= \hat{F}[\ell] = \hat{F}\left[\frac{k}{2}\right]$$

For k odd = $2\ell+1$, $0 \le \ell < \frac{N}{2}$: $\hat{X}[k] = \sum_{n=0}^{\frac{N}{2}-1} x[n] y_N^{-n(2\ell+1)} + \sum_{n=\frac{N}{2}}^{N} x[n] y_N^{-n(2\ell+1)}$

$$\hat{X}[k] = \sum_{n=0}^{N-1} x[n] y_N^{-nk} = \sum_{n=0}^{N-1} x[n] y_N^{-n(2\ell+1)}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x[n] y_N^{-n(2\ell+1)} + \sum_{n=\frac{N}{2}}^{N-1} x[n] y_N^{-n(2\ell+1)}$$

$$= \sum_{m=0}^{\frac{N}{2}-1} x[m] y_N^{-m(2\ell+1)} + \sum_{m=0}^{\frac{N}{2}-1} x\left[m+\frac{N}{2}\right] y_N^{-(m+M)(2\ell+1)}$$

$$= \sum_{m=0}^{M-1} x[m] y_N^{-(2m\ell+m)} + \sum_{m=0}^{M-1} x[m+M] y_N^{-(2m\ell+m+2M\ell+M)}$$

$$= \sum_{m=0}^{M-1} x[m] y_M^{-m\ell} y_N^{-m} + \sum_{m=0}^{M-1} x[m+M] y_M^{-m\ell} y_N^{-m} (1)(-1)$$

$$= \sum_{m=0}^{M-1} \left(y_N^{-m}\left(x[m] - x[m+M]\right)\right) y_M^{-m\ell} = \sum_{m=0}^{M-1} g[m] y_M^{-m\ell}$$

$$= \hat{G}[\ell] = \hat{G}\left[\frac{k-1}{2}\right]$$

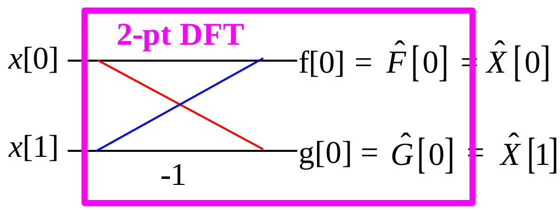
- O($N \log_2 N$)
- Butterfly diagram

- N = 2

$$f[0] = x[0] + x\left[0 + \frac{2}{2}\right] = x[0] + x[1]$$

$$g[0] = y_2^{-0}\left(x[0] - x\left[0 + \frac{2}{2}\right]\right) = x[0] - x[1]$$

$$\begin{bmatrix} \hat{X}[0] \\ \hat{X}[1] \end{bmatrix} = \begin{bmatrix} \hat{F}\left[\frac{0}{2}\right] \\ \hat{G}\left[\frac{1-1}{2}\right] \end{bmatrix} = \begin{bmatrix} \hat{F}[0] \\ \hat{G}[0] \end{bmatrix}$$

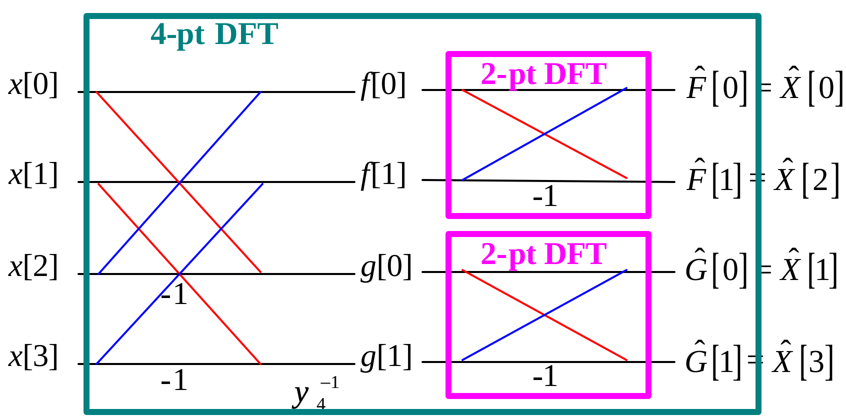


- N = 4

$$\begin{bmatrix} f[0] \\ f[1] \end{bmatrix} = \begin{bmatrix} x[0] + x[0+2] \\ x[1] + x[1+2] \end{bmatrix} = \begin{bmatrix} x[0] + x[2] \\ x[1] + x[3] \end{bmatrix}$$

$$\begin{bmatrix} g[0] \\ g[1] \end{bmatrix} = \begin{bmatrix} (x[0] - x[0+2])y_4^{-0} \\ (x[1] - x[1+2])y_4^{-1} \end{bmatrix} = \begin{bmatrix} x[0] - x[2] \\ (x[1] - x[3])y_4^{-1} \end{bmatrix}$$

$$\begin{bmatrix} \hat{X}[0] \\ \hat{X}[1] \\ \hat{X}[2] \\ \hat{X}[3] \end{bmatrix} = \begin{bmatrix} \hat{F}[0] \\ \hat{G}[0] \\ \hat{F}[1] \\ \hat{G}[1] \end{bmatrix}$$

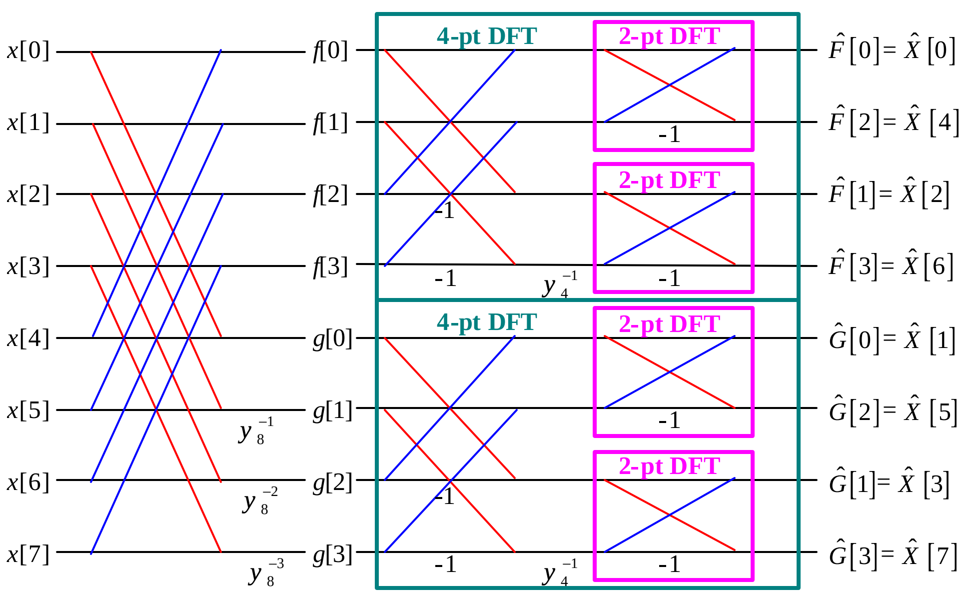


- N = 8

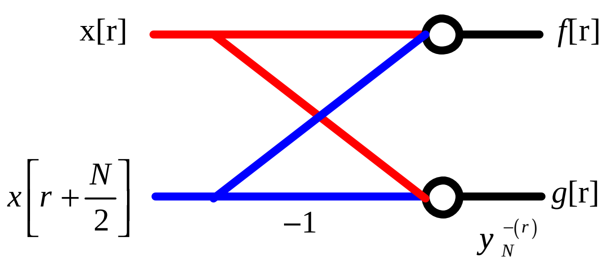
$$\begin{bmatrix} f[0] \\ f[1] \\ f[2] \\ f[3] \end{bmatrix} = \begin{bmatrix} x[0]+x[0+4] \\ x[1]+x[1+4] \\ x[2]+x[2+4] \\ x[3]+x[3+4] \end{bmatrix} = \begin{bmatrix} x[0]+x[4] \\ x[1]+x[5] \\ x[2]+x[6] \\ x[3]+x[7] \end{bmatrix}$$

$$\begin{bmatrix} g[0] \\ g[1] \\ g[2] \\ g[3] \end{bmatrix} = \begin{bmatrix} \left(x[0]-x[0+4]\right)y_8^{-0} \\ \left(x[1]-x[1+4]\right)y_8^{-1} \\ \left(x[2]-x[2+4]\right)y_8^{-2} \\ \left(x[3]-x[3+4]\right)y_8^{-3} \end{bmatrix} = \begin{bmatrix} x[0]-x[4] \\ \left(x[1]-x[5]\right)y_8^{-1} \\ \left(x[2]-x[6]\right)y_8^{-2} \\ \left(x[3]-x[7]\right)y_8^{-3} \end{bmatrix}$$

$$\begin{bmatrix} \hat{X}[0] \\ \hat{X}[1] \\ \hat{X}[2] \\ \hat{X}[3] \\ \hat{X}[4] \\ \hat{X}[5] \\ \hat{X}[6] \\ \hat{X}[7] \end{bmatrix} = \begin{bmatrix} \hat{F}[0] \\ \hat{G}[0] \\ \hat{F}[1] \\ \hat{G}[1] \\ \hat{F}[2] \\ \hat{G}[2] \\ \hat{F}[3] \\ \hat{G}[3] \end{bmatrix}$$



- Typical pair computation:



Goertzel's Algorithm

- $q[n] = \psi_N^{k} q[n-1] + x[n]$

  $q[-1] = 0$

  $\Rightarrow q[N] = \hat{X}[k]$

- #computation = N (still) for each k
- only need to store one number: ψ_N^{-k} (instead of $\forall$k)
- Good for finding $\hat{X}[k]$ for a specific k-value.

  Proof

$$q[0] = x[0]$$

$$q[1] = \psi_N^{k} x[0] + x[1]$$

$$q[2] = \psi_N^{k}\left(\psi_N^{k} x[0] + x[1]\right) + x[2] = \psi_N^{2k} x[0] + \psi_N^{k} x[1] + x[2]$$

$$q[N] = \sum_{\ell=0}^{N} \psi_N^{(N-\ell)k} x[\ell] = \sum_{\ell=0}^{N} \psi_N^{-\ell k} x[\ell] + \cancel{x[N]} = \hat{X}[k]$$