DFT: Discrete Fourier Transform

Congruence (Integer modulo m)

- In this section, all letters stand for integers.
- $\gcd(n,m)$ = the greatest common divisor of n and m
- Let $d = \gcd(n,m)$
  All the linear combinations $r\cdot n + s\cdot m$ of n and m are multiples of d.
- $a|b \Leftrightarrow a$ is a divisor of b.
- In an expression mod m, m is a strictly positive integer.
- $a = b \bmod m \Leftrightarrow m \mid b-a$
- If $a = b \bmod m$
  $c = d \bmod m$
  then
  $$a + c = (b+d) \bmod m$$
  $$a - c = (b-d) \bmod m$$
  $$a\cdot c = (b\cdot d) \bmod m$$
  $$a^n = b^n \bmod m \quad \text{with } n > 0$$
  $$n\cdot a = (n\cdot b) \bmod m$$
- $\langle m\rangle_N = m \bmod N = m + rN$ for unique r that make $0 \le m + rN < N$

N-point signal

- x[n]: **N-point signal** $\Leftrightarrow$
  - Has finite duration
  - Duration interval $\subset$ [0,N)

- **Cyclic shift** of $x[n]$ by n_0 ; $0 \le n < N$; $0 \le n_0 < N$:

$$= x\left[\langle n-n_0\rangle_N\right] = \begin{cases} x[n-n_0] & ;n_0 \le n < N \\ x[N+n-n_0] & ;0 \le n < n_0 \\ 0 & ;otherwise \end{cases}$$

  - $x\left[\langle n-n_0\rangle_N\right]$ is another N-point signal
  - Example
    - $N = 5$

n	-2	-1	0	1	2	3	4	5	6	7
$x[n]$	0	0	a	b	c	d	e	0	0	0

$x\left[\langle n-2\rangle_5\right]$	0	0	d	e	a	E	F		

- **N-point circular convolution** of N-point signal

  $x_1[n] \circledN x_2[n] = \sum_{m=0}^{N-1} x_1[m]x_2\left[\langle n-m\rangle_N\right] = \sum_{m=0}^{N-1} x_1\left[\langle n-m\rangle_N\right]x_2[m]$

Proof

$$\sum_{m=0}^{N-1} x_1\left[\langle n-m\rangle_N\right]x_2[m] = \sum_{m=0}^{n} x_1[n-m]x_2[m] + \sum_{m=n+1}^{N-1} x_1[N+n-m]x_2[m]$$

$$= \sum_{\ell=0}^{n} x_1[\ell]x_2[n-\ell] + \sum_{\ell=n+1}^{N-1} x_1[\ell]x_2[n+N-\ell]$$

$$= \sum_{\ell=0}^{n} x_1[\ell]x_2\left[\langle n-\ell\rangle_N\right] + \sum_{\ell=n+1}^{N-1} x_1[\ell]x_2\left[\langle n-\ell\rangle_N\right]$$

- If $x_1[m]$ has many 0's, use $\sum_{m=0}^{N-1} x_1[m]x_2\left[\langle n-m\rangle_N\right]$, eliminating each of $x_2\left[\langle n-m\rangle_N\right]$ that is multiplied by $x_1[m]=0$
- Example:
  - N = 3

    Let $y[n] = x_1[n] \circledN x_2[n]$

    $$y[n] = \sum_{m=0}^{2} x_1[m]x_2\left[\langle n-m\rangle_3\right] = x_1[0]x_2[n] + x_1[1]x_2\left[\langle n-1\rangle_3\right] + x_1[2]x_2\left[\langle n-2\rangle_3\right]$$

    $$y[0] = x_1[0]x_2[0] + x_1[1]x_2\left[\langle -1\rangle_3\right] + x_1[2]x_2\left[\langle -2\rangle_3\right]$$

    $$= x_1[0]x_2[0] + x_1[1]x_2[2] + x_1[2]x_2[3]$$

    $$y[1] = x_1[0]x_2[1] + x_1[1]x_2\left[\langle 0\rangle_3\right] + x_1[2]x_2\left[\langle -1\rangle_3\right]$$

    $$= x_1[0]x_2[1] + x_1[1]x_2[0] + x_1[2]x_2[2]$$

    $$y[2] = x_1[0]x_2[2] + x_1[1]x_2\left[\langle 1\rangle_3\right] + x_1[2]x_2\left[\langle 0\rangle_3\right]$$

    $$= x_1[0]x_2[2] + x_1[1]x_2[1] + x_1[2]x_2[0]$$

- To find $x_1[n] \circledN x_2[n]$ using circular convolution rule
  - $\hat{X}_1[k] = x_1[0] + x_1[1]y_3^{-k} + x_1[2]y_3^{-2k} = x_1[0] + x_1[1]A + x_1[2]A^2$

    $\hat{X}_2[k] = x_2[0] + x_2[1]y_3^{-k} + x_2[2]y_3^{-2k} = x_2[0] + x_2[1]A + x_2[2]A^2$

    The multiplication $\hat{X}_1[k]\cdot\hat{X}_2[k]$ can be easily find with TI calculator or Mathcad (select A, then choose Symbolics > Polynomial Coefficients):

$$\hat{X}_1[k]\cdot\hat{X}_2[k]=\begin{bmatrix}1 & A & A^2 & A^3 & A^4\end{bmatrix}\begin{bmatrix}x_{10}x_{20}\\ x_{10}x_{21}+x_{11}x_{20}\\ x_{10}x_{22}+x_{11}x_{21}+x_{12}x_{20}\\ x_{11}x_{22}+x_{12}x_{21}\\ x_{12}x_{22}\end{bmatrix}$$

But $A^3=1, A^4=A$; therefore,

$$\hat{X}_1[k]\cdot\hat{X}_2[k]=\begin{bmatrix}1 & A & A^2\end{bmatrix}\begin{bmatrix}x_{10}x_{20}+x_{11}x_{22}+x_{12}x_{21}\\ x_{10}x_{21}+x_{11}x_{20}+x_{12}x_{22}\\ x_{10}x_{22}+x_{11}x_{21}+x_{12}x_{20}\end{bmatrix}.$$

Obviously (actually from inverse DFT), $\begin{bmatrix}y[0]\\ y[1]\\ y[2]\end{bmatrix}=\begin{bmatrix}x_{10}x_{20}+x_{11}x_{22}+x_{12}x_{21}\\ x_{10}x_{21}+x_{11}x_{20}+x_{12}x_{22}\\ x_{10}x_{22}+x_{11}x_{21}+x_{12}x_{20}\end{bmatrix}$.

- Example:
  - $x_1[n]$ = 2 3 1 1, $x_2[n]$ = 1 0 1 2

$$\hat{X}_1[k]\cdot\hat{X}_2[k]=\left(2+3y_4^{-k}+1y_4^{-2k}+1y_4^{-3k}\right)\cdot\left(1+0y_4^{-k}+1y_4^{-2k}+2y_4^{-3k}\right)$$
$$=2+3y_4^{-k}+3y_4^{-2k}+8y_4^{-3k}+8y_4^{-4k}+3y_4^{-5k}+2y_4^{-6k}$$
$$=10+6y_4^{-k}+5y_4^{-2k}+8y_4^{-3k}$$

$x_1[n]$ Ⓝ $x_2[n]$ = 10 6 5 8

DFT

- $y_N=e^{j\frac{2p}{N}}$
  - $(y_N)^N=1$
  - $\overline{y_N}=(y_N)^{-1}$
  - $N \bmod p=0 \Rightarrow y_N^p=y_{\frac{N}{p}}$
    - $y_{kN}^N=y_k$
  - For even n , $y_N^{\frac{N}{2}}=e^{jp}=-1$
  - $\sum_{n=0}^{N-1}y_N^{-kn}=\begin{cases}N & k=mN\\ 0 & otherwise\end{cases}$

Proof $(y_N)^N=\left(e^{j\frac{2p}{N}}\right)^N=e^{j2p}=1$

Proof $\overline{y_N} = \overline{e^{j\frac{2p}{N}}} = e^{-j\frac{2p}{N}} = \left(e^{j\frac{2p}{N}}\right)^{-1}$

Proof $N \bmod p = 0 \Rightarrow y_N^{\,p} = \left(e^{\frac{2p}{N}}\right)^{p} = e^{\frac{2p}{N/p}} = y_{\frac{N}{p}}$

Proof $\sum_{n=0}^{N-1} e^{j2p\,n\frac{k}{N}} = \begin{cases} N\ ;\ \text{if } \dfrac{k}{N} \in I \\ 0\ ;\ \text{if } \dfrac{k}{N} \notin I \end{cases}$

$$\sum_{n=0}^{N-1} y_N^{\,-kn} = \sum_{n=0}^{N-1} e^{j\frac{2p}{N}(-kn)} = \sum_{n=0}^{N-1} e^{j\frac{2p}{N}nk} = \begin{cases} N\ ;\ \text{if } \dfrac{k}{N} \in I \\ 0\ ;\ \text{if } \dfrac{k}{N} \notin I \end{cases}$$

- $\left[\Psi_N\right]_{pq} = y_N^{\,-(p-1)(q-1)}\ ;\ 1 \le p,q \le N$

- $\Psi_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & y_N^{-1} & y_N^{-2} & \cdots & y_N^{-(N-1)} \\ 1 & y_N^{-2} & y_N^{-4} & \cdots & y_N^{-2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & y_N^{-(N-1)} & y_N^{-2(N-1)} & \cdots & y_N^{-(N-1)(N-1)} \end{bmatrix}$

- $\Psi_N^{-1} = \dfrac{1}{N}\overline{\Psi_N}$

Proof

$$\begin{aligned} \left[\overline{\Psi_N}\Psi_N\right]_{pq} &= \sum_{\ell=1}^{N}\left[\overline{\Psi_N}\right]_{p\ell}\left[\Psi_N\right]_{\ell q} = \sum_{\ell=1}^{N} y_N^{\,(p-1)(\ell-1)} y_N^{\,-(\ell-1)(q-1)} \\ &= \sum_{\ell=1}^{N} y_N^{\,(\ell-1)(p-q)} = \sum_{m=0}^{N-1} y_N^{\,m(p-q)} \\ &= \begin{cases} N & ;\ p = q \Rightarrow y_N^{\,p-q} = 1 \\ \dfrac{1-\overbrace{y_N^{\,N(p-q)}}^{1}}{1-y_N^{\,(p-q)}} = 0 & ;\ p \ne q \end{cases} \\ &= NI_N \end{aligned}$$

- **N-point DFT** of the N-point signal x[n]:

  $$\hat{X}[k] = \hat{X}\left(w = k\frac{2p}{N}\right) = \sum_{n=0}^{N-1} x[n]\, y_N^{\,-nk}\ ;\ 0 \le k < N$$

- $\hat{X}[k] = \hat{X}\left(w = k\frac{2p}{N}\right) = \sum_{n=0}^{N-1} x[n] e^{-jnk\frac{2p}{N}} = \sum_{n=0}^{N-1} x[n] y_N^{-nk} \quad ;0 \le k < N$

- $$\begin{bmatrix} \hat{X}[0] \\ \hat{X}[1] \\ \hat{X}[2] \\ \vdots \\ \hat{X}[n] \end{bmatrix} = \Psi_N \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[n] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & y_N^{-1} & y_N^{-2} & \cdots & y_N^{-(N-1)} \\ 1 & y_N^{-2} & y_N^{-4} & \cdots & y_N^{-2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & y_N^{-(N-1)} & y_N^{-2(N-1)} & \cdots & y_N^{-(N-1)(N-1)} \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[n] \end{bmatrix}$$

- $d[n] \overset{DFT}{\rightleftharpoons} \sum_{n=0}^{N-1} d[n] y_N^{-nk} = y_N^{-0k} = 1$

- $\boxed{\left(DFT^{-1}\right) \underset{0\le n<N}{x[n]} = \frac{1}{N}\sum_{k=0}^{N-1} \hat{X}[k] y_N^{nk} \overset{DFT}{\rightleftharpoons} \underset{0\le k<N}{\hat{X}[k]} = \sum_{n=0}^{N-1} x[n] y_N^{-nk} \quad (DFT)}$

Proof $\underline{x} = \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[n] \end{bmatrix}$; $\underline{\hat{X}} = \begin{bmatrix} \hat{X}[0] \\ \hat{X}[1] \\ \hat{X}[2] \\ \vdots \\ \hat{X}[n] \end{bmatrix}$

$$\Rightarrow \underline{\hat{X}} = \Psi_N \underline{x} \Rightarrow \underline{x} = \frac{1}{N}\Psi_N^* \underline{\hat{X}}$$

- Given tool for computing DFT

$$\underset{0\le k<N}{\hat{X}[k]} \rightarrow \underbrace{\boxed{\boxed{DFT} \rightarrow \boxed{\text{Time reverse}} \rightarrow \boxed{\times\frac{1}{N}}}}_{DFT^{-1}} \rightarrow \underset{0\le k<N}{x[n]}$$

Proof After DFT-finder g[q] = $\sum_{p=0}^{N-1} \hat{X}[p] y_N^{-pq}$,

Time reverse: g[N-q] = $\sum_{p=0}^{N-1} \hat{X}[p] y_N^{-p(N-q)} = \sum_{p=0}^{N-1} \hat{X}[p] y_N^{pq}$

- $\boxed{\hat{X}[k] \xrightarrow{DFT^{-1}} x[n] \xrightarrow{DTFT} \hat{X}(w)}$

- $\boxed{\underset{0\le n<N}{x[\langle n-n_0\rangle]} \overset{DFT}{\rightleftharpoons} \underset{0\le k<N}{y_N^{-kn_0}\hat{X}[k]}}$

Proof

$$\sum_{n=0}^{N-1} x\left[\langle n-n_0\rangle_N\right] y_N^{-nk} = \sum_{n=n_0}^{N-1} x[n-n_0] y_N^{-nk} + \sum_{n=0}^{n_0-1} x[N+n-n_0] y_N^{-nk}$$

$$= \sum_{\ell=0}^{N-n_0-1} x[\ell] y_N^{-(\ell+n_0)k} + \sum_{\ell=N-n_0}^{N-1} x[\ell] y_N^{-(\ell+n_0-\cancel{N})k}$$

$$= \sum_{\ell=0}^{N-1} x[\ell] y_N^{-(\ell+n_0)k}$$

- $d[n-n_0] \xleftrightarrow{DFT} y_N^{-n_0 k}$; $0 \le n_0 < N$

Proof $d\left[\langle n-n_0\rangle_N\right] = \begin{cases} d[n-n_0] & ; n_0 \le n < N \\ d[N+n-n_0] & ; 0 \le n < n_0 \\ 0 & ; otherwise \end{cases} = d[n-n_0]$

$$\underset{0\le n<N}{d\left[\langle n-n_0\rangle\right]} \xrightarrow{DFT} \underset{0\le k<N}{y_N^{-kn_0} \times 1}$$

- **Circular convolution rule**:

$x_1[n] \circledN x_2[n] \xrightleftharpoons{DFT} \hat{X}_1[k] \cdot \hat{X}_2[k]$

Proof

$$\sum_{n=0}^{N-1}\left(\sum_{m=0}^{N-1} x_1[m] x_2\left[\langle n-m\rangle_N\right]\right) y_N^{-nk} = \sum_{m=0}^{N-1} x_1[m]\left(\sum_{n=0}^{N-1} x_2\left[\langle n-m\rangle_N\right] y_N^{-nk}\right)$$

$$= \sum_{m=0}^{N-1} x_1[m]\left(\hat{X}[k] y_N^{-mk}\right)$$

- To use DFT to compute regular convolutions of time-limited (finite-duration) signal:

If $x_1[n]$ has duration interval $0 \le n < N_1$

$x_2[n]$ has duration interval $0 \le n < N_2$

Let $N = N_1+N_2-1$,

then $x_1[n]*x_2[n] = x_1[n] \circledN x_2[n]$ for $0 \le n < N$ and $= 0$ otherwise

To see this,

$x_1[n]*x_2[n]$ has finite duration at most $N = N_1+N_2-1$

Think of $x_1[n]$ and $x_2[n]$ as N-point signal whose last values are 0 (0-padding)

- **Block convolution**

$h[n]$ has duration interval $0 \le n < P$ (impulse response of a causal FIR system)

$w[n]$ = a (possibly) infinite-duration signal

To compute $h[n]*w[n]$

  - Divide $w[n]$ into blocks of some specific length L (typically $L >> P$)

$w_r[n] = \begin{cases} w[rL+n] & ; 0 \le n < L \\ 0 & ; otherwise \end{cases}$ duration interval $0 \le n < L$

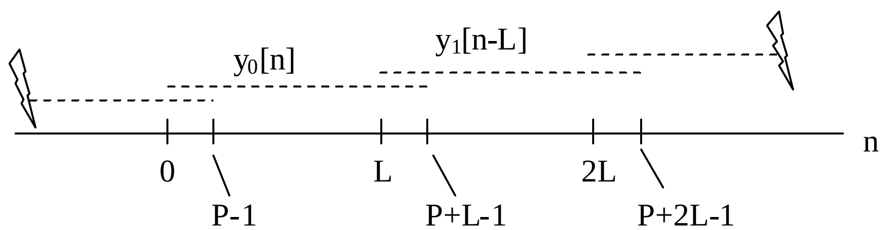
$$w[n] = \sum_{r=-\infty}^{\infty} w_r[n-rL] \quad ,\forall n$$

- $y_r[n] = h[n] * w_r[n]$ duration interval $0 \le n < P + L - 1$
  Find $y_r[n]$ by $(P+L-1)$-point circular convolution
- $h[n] * w[n] = \sum_{r=-\infty}^{\infty} y_r[n-rL] \quad ,\forall n$

Proof $\quad h[n]*w[n] = h[n]*\sum_{r=-\infty}^{\infty} w_r[n-rL] = \sum_{r=-\infty}^{\infty} h[n]*w_r[n-rL]$

$$= \sum_{r=-\infty}^{\infty} y_r[n-rL]$$

Note that $h[n]*w_r[n-rL] = y_r[n-rL]$ from the time-invariance property of convolution.



- **Frequency-sampling approximation** of h[n]
  To approximate $h[n]$:
  - Sampling $\hat{H}(w)$ at $w = k\frac{2p}{N}$; $0 \le k < N$ by
    $e^{jk\frac{2p}{N}n} \xrightarrow{h[n]}$ yield $\tilde{\hat{H}}[k] = \hat{H}\left(w = k\frac{2p}{N}\right)$
  - Use $(\text{DFT})^{-1}$, $\tilde{h}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{\hat{H}}[k]y_N^{nk} = \sum_{r=-\infty}^{\infty} h[n-rN]$; $0 \le n < N$

$$\frac{\left(\hat{H}(w)\big|_{w=w_0}\right)e^{jnw_0}}{e^{jnw_0}} = \left(\hat{H}(w)\big|_{w=w_0}\right)$$

$$\tilde{\hat{H}}[k] = \hat{H}\left(w = k\frac{2p}{N}\right) \xrightarrow{DFT^{-1}} \tilde{h}[n] = \begin{cases} \sum_{r=-\infty}^{\infty} h[n-rN] & 0 \le n < N \\ 0 & \text{otherwise} \end{cases}$$

- $\underset{0\le n<N}{y[n]} = \sum_{r=-\infty}^{\infty} x[n-rN] \underset{}{\overset{DFT}{\rightleftharpoons}} \underset{0\le k<N}{\hat{Y}[k]} = \hat{X}\left(w = k\frac{2p}{N}\right)$

Proof

$$\tilde{h}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{\hat{H}}[k]\psi_N^{nk} = \frac{1}{N}\sum_{k=0}^{N-1}\hat{H}\left(k\frac{2\pi}{N}\right)\psi_N^{nk}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}\left(\sum_{m=-\infty}^{\infty} h[m]e^{-jmk\frac{2\pi}{N}}\right)e^{j\frac{2\pi}{N}nk}$$

$$= \sum_{m=-\infty}^{\infty} h[m]\left(\frac{1}{N}\sum_{k=0}^{N-1}e^{j\frac{2\pi}{N}k(n-m)}\right)$$

$$\frac{1}{M}\sum_{\ell=0}^{M-1}e^{j2\pi\ell\frac{n}{M}} = \begin{cases}1; \text{ if } \frac{n}{M}\in I\\ 0; \text{ if } \frac{n}{M}\notin I\end{cases} \Rightarrow \frac{1}{N}\sum_{k=0}^{N-1}e^{j2\pi k\frac{n-m}{N}} = \begin{cases}1; \text{ if } \frac{n-m}{N}\in I\\ 0; \text{ if } \frac{n-m}{N}\notin I\end{cases}$$

$$\tilde{h}[n] = \sum_{\substack{m=-\infty\\ \frac{n-m}{N}\in I}}^{\infty} h[m]$$

So, the summation only include m of the form $m = n - rN$; $r \in I$

Thus, $\tilde{h}[n] = \sum_{r=-\infty}^{\infty} h[n-rN]$

Proof 2: $\tilde{h}[n] = \sum_{r=-\infty}^{\infty} h[n-rN] \xrightarrow{DFT} \tilde{\hat{H}}[k] = \hat{H}\left(\omega = k\frac{2\pi}{N}\right)$

First, note that $\hat{H}\left(k\frac{2\pi}{N}\right) = \sum_{n=-\infty}^{\infty} h[n]e^{-jnk\frac{2\pi}{N}} = \sum_{n=-\infty}^{\infty} h[n]\psi_N^{-nk}$.

$$\tilde{\hat{H}}[k] = \sum_{n=0}^{N-1}\tilde{h}[n]\psi_N^{-nk} = \sum_{n=0}^{N-1}\left(\sum_{r=-\infty}^{\infty} h[n+rN]\right)\psi_N^{-nk}$$

$$= \sum_{r=-\infty}^{\infty}\sum_{n=0}^{N-1}\left(h[n+rN]\psi_N^{-nk}\right)$$

Substitute $\ell = n + rN \Rightarrow n = \ell - rN$

$$\tilde{\hat{H}}[k] = \sum_{r=-\infty}^{\infty}\sum_{\ell=rN}^{(r+1)N-1}\left(h[\ell]\psi_N^{-(\ell-rN)k}\right) = \sum_{r=-\infty}^{\infty}\sum_{\ell=rN}^{(r+1)N-1}\left(h[\ell]\psi_N^{-\ell k}\right)$$

Note that $\sum_{r=-\infty}^{\infty}\sum_{\ell=rN}^{(r+1)N-1} x(\ell) = \sum_{r=-\infty}^{\infty}\sum_{\ell=rN}^{rN+(N-1)} x(\ell) = \sum_{\ell=-\infty}^{\infty} x[\ell]$.

Thus, $\tilde{\hat{H}}[k] = \sum_{\ell=-\infty}^{\infty} h[\ell]\psi_N^{-\ell k} = \hat{H}\left(k\frac{2\pi}{N}\right)$.

- Time-aliasing
- If $h[n]$ has duration interval contained in $0 \le n < N$,
  $\tilde{h}[n] = h[n]$; $0 \le n < N$
- Example: if $h[n]$ is a (2N)-point signal

$$\hat{H}\left(k\frac{2p}{N}\right)=\sum_{n=-\infty}^{\infty}h[n]e^{-jnk\frac{2p}{N}}=\sum_{n=-\infty}^{\infty}h[n]y_N^{-nk}=\sum_{n=0}^{2N-1}h[n]y_N^{-nk}$$

$$=\sum_{n=0}^{N-1}h[n]y_N^{-nk}+\sum_{n=N}^{2N-1}h[n]y_N^{-nk}$$

$$=\sum_{n=0}^{N-1}h[n]y_N^{-nk}+\sum_{\ell=0}^{N-1}h[\ell+N]y_N^{-(\ell+N)k}$$

$$=\sum_{n=0}^{N-1}h[n]y_N^{-nk}+\sum_{\ell=0}^{N-1}h[\ell+N]y_N^{-\ell k}$$

$$=\sum_{n=0}^{N-1}(h[n]+h[n+N])y_N^{-nk}$$

$$\tilde{h}[n]=h[n]+h[n+N]\ ;0\le n<N$$

So, $\tilde{h}[n]=h[n]$; $0\le n<N$ iff $h[n]=0$ for $n\ge N$ (no folding)

- **Windowing a signal** to get an approximation of $\hat{X}(w)$

  Given $x[n]$ $-\infty<n<\infty$

  Look at $y[n]=x[n]p_L[n]=\begin{cases}x[n] & ;-L<n<L\\ 0 & ;|n|\ge L\end{cases}$ (length = $2L$-1 = N)

  Find $\hat{Y}(w)=\dfrac{1}{2p}\displaystyle\int_{-p}^{p}\frac{\sin\left(\left(L-\frac{1}{2}\right)m\right)}{\sin\left(\frac{m}{2}\right)}\hat{X}(w-m)dm$

- $p_L[n]=\left.\begin{cases}1 & ;-L<n<L\\ 0 & ;|n|\ge L\end{cases}\right\}\overset{DTFT}{\rightleftharpoons}\dfrac{\sin\left(\left(L-\frac{1}{2}\right)w\right)}{\sin\left(\frac{w}{2}\right)}$

  - Zeros @ $\dfrac{kp}{L-\frac{1}{2}}$
  - = $2L$-1 @ $w\to 0$

- Need width of $\hat{X}(w)$'s feature > $\dfrac{2p}{L-\frac{1}{2}}=\dfrac{4p}{N}$

- If $L\to\infty$, $\lim_{L\to\infty}p_L[n]=1\overset{DTFT}{\rightleftharpoons}2pd(w)$, and $\hat{Y}(w)\to\hat{X}(w)$
- Lots of "sharp activity" in $\hat{X}(w)\Rightarrow$ need bigger N

- Given $x[n]$; $0\le n<N$

Can use M-point DFT's to find $\hat{X}[k]$ only for k = $\frac{N}{M}\ell$ = rℓ ; $0 \le \ell < M$; $r = \frac{N}{M} \in I$

by let $y[n]$ = $\sum_{p=0}^{r-1} x[n+(p-1)M]$, then $\hat{X}[k] = \hat{Y}[\ell]$

Proof

$$\hat{X}[k] = \sum_{n=0}^{N-1} x[n] y_N^{-nk} = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2p}{N}\left(\frac{N}{M}\ell\right)n} = \sum_{n=0}^{N-1} x[n] y_M^{-\ell n}$$

$$= \sum_{p=0}^{r-1} \sum_{n=pM}^{(p+1)M-1} x[n] y_M^{-\ell n} = \sum_{p=0}^{r-1} \sum_{n=0}^{M-1} x[n+(p-1)M] y_M^{-\ell(n+(p-1)M)}$$

$$= \sum_{n=0}^{M-1} \sum_{p=0}^{r-1} x[n+(p-1)M] y_M^{-\ell n}$$

- Let x[n] be a signal of duration N.
  If already have tool for M-point DFT's.

  Can find $\hat{X}(w)$ for $\omega_k = \omega_0 + k\frac{2p}{M}$, by constructing y[n] so $\hat{Y}[k] = \hat{X}(w_k)$.

  To do this,

  - Let q[n] = $e^{-jw_0 n}$ x[n], then $\hat{Q}(w) = \hat{X}(w + w_0)$ and

    $\hat{Q}\left(k\frac{2p}{M}\right) = \hat{X}\left(k\frac{2p}{M} + w_0\right) = \hat{X}(w_k)$

  - Want $\hat{Y}[k] = \hat{Q}\left(k\frac{2p}{M}\right)$,

    From $y[n] = \sum_{r=-\infty}^{\infty} x[n - rN] \overset{DFT}{\rightleftharpoons} \hat{Y}[k] = \hat{X}\left(w = k\frac{2p}{N}\right)$, so need

    $$y[n] = \begin{cases} \sum_{r=-\infty}^{\infty} q[n + rM] & 0 \le n < M \\ 0 & otherwise \end{cases}$$

  - Note that not all *r*'s are used. only ones that satisfy

    0 < n+rM < N $\Rightarrow -\frac{n}{M} \le r < \frac{N-n}{M}$, but 0 ≤ n < M. So, -1 < r < $\frac{N}{M}$

    $\Rightarrow$ 0 ≤ r < $\frac{N}{M}$