Discrete-time Fourier Transform

- To distinguish between the frequency domains of DTFT and CTFT, we will use Ω for CT and w for DT.

- Discrete-time Fourier Transform (DTFT)

$$x[n] = \frac{1}{2p}\int_{-p}^{p} \hat{X}(w) e^{jnw} dw \xrightleftharpoons{DTFT} \hat{X}(w) = \sum_{n=-\infty}^{\infty} x[n] e^{-jnw}$$

- Integration: [-π,π) or (-π,π]

Proof $e^{jnw}\hat{X}(w) = \sum_{m=-\infty}^{\infty} x[m] e^{-j(m-n)w}$

$$= \ldots + x[n-1]e^{-j(-1)w} + x[n]e^{-j(0)w} + x[n+1]e^{-j(1)w} + \ldots$$

Since $\int_{2p} e^{-jkw} dq = \begin{cases} 2p & k = 0 \\ \frac{1}{-jk}\left(e^{-jk(a+2p)} - e^{-jka}\right)^{0} & k \neq 0 \end{cases} = \begin{cases} 2p & k = 0 \\ 0 & k \neq 0 \end{cases}$,

$$\int_{-p}^{p} e^{jnw}\hat{X}(w) dw = \ldots + 0 + 2p\, x[n] + 0 + \ldots = 2p\, x[n]$$

- $\hat{X}(w + 2kp) = \hat{X}(w) \Rightarrow$ Therefore, consider only on $-p < \omega < p$

Proof $e^{-jn(w+2kp)} = e^{-jnw}\left(e^{-j2p}\right)^{nk} = e^{-jnw}$ (with e^{-j2p} struck through and replaced by 1)

$x[n]$	$\hat{X}(w) = \sum_{k=-\infty}^{\infty}$ of
Linearity: $\sum x_i[n]$	$\sum \hat{X}_i(w)$
Time-shift: $x[n - n_0]$	$e^{-jn_0 w}\hat{X}(w)$

Proof $\sum_{n=-\infty}^{\infty} x[n-n_0] e^{-jnw} = \sum_{m=-\infty}^{\infty} x[m] e^{-j(m+n_0)w}$; $m = n - n_0$

$$= e^{-jn_0 w}\left(\sum_{m=-\infty}^{\infty} x[m] e^{-jmw}\right)$$

Frequency-shift: $e^{jnw_0}x[n]$	$\hat{X}(w - w_0)$

Proof $\hat{X}(w - w_0) = \sum_{n=-\infty}^{\infty}\left(x[n] e^{-jn(w-w_0)}\right) = \sum_{n=-\infty}^{\infty}\left(\underbrace{e^{jnw_0} x[n]} e^{-jnw}\right)$

Convolution in time: $x_1[n] * x_2[n]$	$\hat{X}_1(w)\cdot\hat{X}_2(w)$

Proof

$$\sum_{n=-\infty}^{\infty}\left(\sum_{m=-\infty}^{\infty}\left(x_1[m]\,x_2[n-m]\right)\right)e^{-jnw} = \sum_{m=-\infty}^{\infty}\left(x_1[m]\sum_{n=-\infty}^{\infty}\left(x_2[n-m]e^{-jnw}\right)\right)$$

$$= \sum_{m=-\infty}^{\infty}\left(x_1[m]e^{-jmw}\hat{X}_2(w)\right) \text{ ; time-shift rule}$$

Convolution in frequency: $x_1[n]\cdot x_2[n]$	$\underbrace{\frac{1}{2p}\int_{2p}\hat{X}_1(m)\hat{X}_2(w-m)dm}_{\text{periodic convolution}}$

Proof $\frac{1}{2p}\int_{2p}\left\{\frac{1}{2p}\int_{2p}\hat{X}_1(m)\hat{X}_2(w-m)dm\right\}e^{jnw}dw$

$$=\frac{1}{2p}\int_{2p}\left\{\hat{X}_1(m)\left\{\frac{1}{2p}\int_{2p}\hat{X}_2(w-m)e^{jnw}dw\right\}\right\}dm$$

$$=\frac{1}{2p}\int_{2p}\left\{\hat{X}_1(m)e^{jnm}x_2[n]\right\}dm \text{ by frequency-shift rule}$$

e^{jnw_0}	$2pd(w-w_0+2kp)$

Proof $\frac{1}{2p}\int_{-p}^{p}(2pd(w-w_0))e^{jnw}dw = e^{jnw_0}$

• $(-1)^n$	$2pd(w-p+2kp)$

Proof Use $(-1)^n = e^{jnp}$.

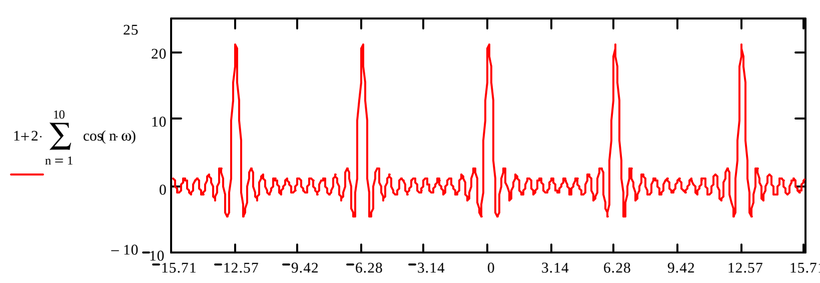
1	$2pd(w+2kp)$

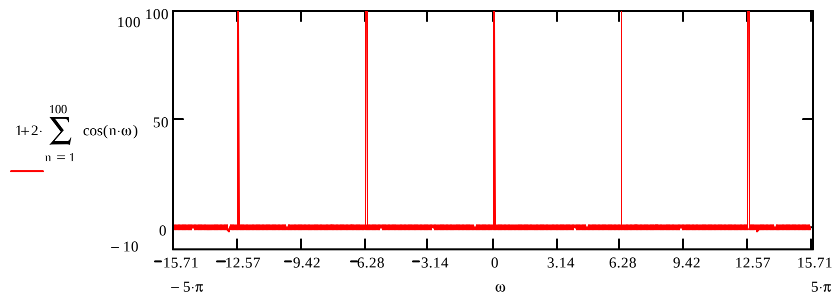
Proof Use $e^{jnw_0}\xrightleftharpoons{DTFT}2pd(w-w_0+2kp)$ and let $w_0=0$

Proof $\hat{X}(w)=\sum_{n=-\infty}^{\infty}1e^{-jnw}=\sum_{n=-\infty}^{\infty}e^{-jnw}$

Can see that when $w=0,2kp \Rightarrow e^{-jnw}\to 1$ and $\sum_{n=-\infty}^{\infty}e^{-jnw}\to\infty$

When $w\neq 0,2kp$, $\sum_{n=-\infty}^{\infty}e^{-jnw}=1+\sum_{n=1}^{\infty}\left(e^{-jnw}+e^{jnw}\right)=1+2\sum_{n=1}^{\infty}\cos(nw)$





$$\int_{-p}^{p} 1+\ 2\sum_{n=1}^{\infty}\cos(nw)dw = 2p$$

$d[n]$	1

Proof $\sum_{n=-\infty}^{\infty} d[n]e^{-jnw} = e^{-j0w} = 1$

Proof $\frac{1}{2p}\int_{-p}^{p} 1e^{jnw}dw = \frac{1}{2p}\int_{-p}^{p} 1e^{jnw}dw = \frac{1}{2p}\frac{1}{jn}e^{jnw}\Big|_{-p}^{p}$

$$= \frac{1}{2p}\frac{1}{jn}\left(e^{jnp} - e^{-jnp}\right) = \frac{\sin(np)}{np}$$

$$\lim_{n\to 0}\frac{\sin(np)}{np} = \lim_{n\to 0}\frac{p\cos(np)}{p} = 1$$

$d[n-n_0]$	e^{-jn_0w}

Proof $\sum_{n=-\infty}^{\infty} d[n-n_0]e^{-jnw} = e^{-jn_0w}$

$z_0^n u[n]$ with $\lvert z_0\rvert < 1$	$\frac{e^{jw}}{e^{jw} - z_0}$

Proof $z_0^n u[n] \overset{Z}{\rightleftharpoons} \frac{z}{z - z_0}$; $(ROC)_X = (\lvert z_0\rvert, \infty)$

$$\hat{X}(w) = X(z)\Big|_{z=e^{jw}} = \frac{z}{z-z_0}\Bigg|_{z=e^{jw}}$$

$u[n]$	$\frac{1}{1-e^{-jw}} + \sum_{k=-\infty}^{\infty} pd(w+2kp)$
$\cos(nw_0)$	$p\left(d(w-w_0+2kp) + d(w+w_0+2kp)\right)$

Proof $\cos(nw_0) = \frac{1}{2}\left(e^{jnw_0} + e^{-jnw_0}\right)$

$\sin(nw_0)$	$\frac{p}{j}\left(d(w-w_0+2kp) - d(w+w_0+2kp)\right)$

Proof $\sin(nw_0) = \frac{1}{2j}\left(e^{jnw_0} - e^{-jnw_0}\right)$

$x[n]\cos(nw_0)$	$\frac{1}{2}\sum_{k=-\infty}^{\infty}\left(\hat{X}(w-\langle w_0\rangle+2kp)+\hat{X}(w+\langle w_0\rangle+2kp)\right)$
$\sum_{k=0}^{M-1} d[n-k]$	$e^{-j\frac{M-1}{2}w}\underbrace{\left(\frac{\sin\left(\frac{M}{2}w\right)}{\sin\left(\frac{1}{2}w\right)}\right)}_{\text{periodic sinc}}$

Proof $\sum_{k=0}^{M-1} d[n-k] \overset{Z}{\rightleftharpoons} \frac{1-z^{-M}}{1-z^{-1}}$; $0<|z|<\infty$

Thus, $1=\left|e^{jw}\right| \in (ROC)_X$ and $\hat{X}(w)=X(z)\big|_{z=e^{jw}}$

$$\hat{X}(w)=\frac{1-e^{-jMw}}{1-e^{-jw}}=\frac{e^{-j\frac{M}{2}w}}{e^{-j\frac{1}{2}w}}\frac{e^{j\frac{M}{2}w}-e^{-j\frac{M}{2}w}}{e^{j\frac{1}{2}w}-e^{-j\frac{1}{2}w}}=e^{-j\frac{M-1}{2}w}\underbrace{\left(\frac{\sin\left(\frac{M}{2}w\right)}{\sin\left(\frac{1}{2}w\right)}\right)}_{\text{periodic sinc}}$$

Proof

$\frac{\sin(nw_c)}{pn}=\underbrace{\frac{w_c}{p}\text{sinc}(nw_c)}_{\text{sampled sinc function}}$	$p_{w_c}(w+2kp)$

Proof $\frac{1}{2p}\int_{-p}^{p} p_{w_c}(w)e^{jnw}dw=\frac{1}{2p}\int_{-w_c}^{w_c} e^{jnw}dw=\frac{1}{2pjn}\left(e^{jnw_c}-e^{-jnw_c}\right)$

Proof

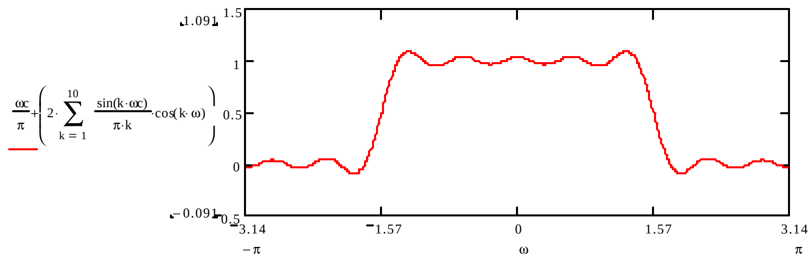
$$\hat{X}(w)=\sum_{n=-\infty}^{\infty} x[n]e^{-jnw}=\sum_{n=-\infty}^{\infty}\frac{\sin(nw_c)}{pn}e^{-jnw}$$

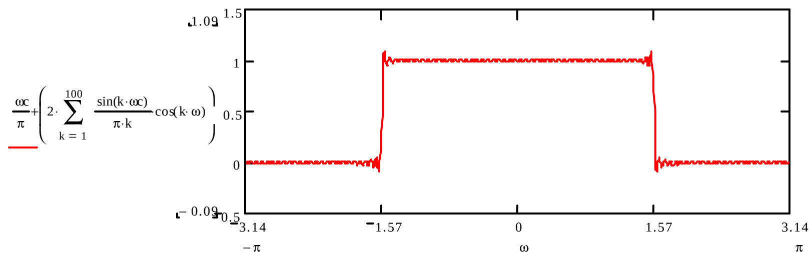
$$=\lim_{n\to 0}\frac{\sin(nw_c)}{pn}+\sum_{k=1}^{\infty}\left(\frac{\sin(kw_c)}{pk}e^{-jkw}+\frac{-\sin(kw_c)}{-pk}e^{jkw}\right)$$

$$=\lim_{n\to 0}\frac{\sin(nw_c)}{pn}+\sum_{k=1}^{\infty}\left(2\frac{\sin(kw_c)}{pk}\cos(kw)\right)$$

$$=\frac{w_c}{p}+\sum_{k=1}^{\infty}\left(2\frac{\sin(kw_c)}{pk}\cos(kw)\right)$$

The figures below shows the summations when k goes as far as 10 and 100, and $w_c=\frac{p}{2}$





- Can see this by expanding $p_{w_C}(w+2kp)$ into its Fourier series.

DTFT and Z-transform

- $\hat{X}(w) = \sum_{n=-\infty}^{\infty} x[n]e^{-jnw}$

  $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$; $(ROC)_X$ $0 \le R_a < |z| < R_b \le \infty$

- If $|z| = 1$ is in $(ROC)_X$,

  $\hat{X}(w) = X(z)\big|_{z=e^{jw}}$

  To see this, $|z| = 1$ is in $(ROC)_X \Rightarrow \left|e^{jw}\right| = 1$ is in $(ROC)_X$

  $$\hat{X}(w) = \left(\sum_{n=-\infty}^{\infty} x[n]z^{-n}\right)\Bigg|_{z=e^{jw}} = X(z)\big|_{z=e^{jw}}$$

- If $|z| = 1$ is bounded away form $(ROC)_X$,
  $x[n]$ has no DTFT
  - Example
    - $x[n] = z_0^n u[n] \overset{Z}{\rightleftharpoons} \frac{z}{z-z_0}$; $(ROC)_X = (|z_0|, \infty)$

      with $|z_0| > 1$

      $\Rightarrow 1 \notin (|z_0|, \infty)$

- If $|z| = 1$ is a bounding circle (edge) for $(ROC)_X$
  $x[n]$ may or may not have a DTFT
  If it does, DTFT usually contains impulses
  - Example

- $u[n] \xrightleftharpoons{DTFT} \frac{1}{1-e^{-jw}} + \sum_{k=-\infty}^{\infty} pd(w+2kp)$

  $x[n] = u[n] \xrightleftharpoons{Z} \frac{z}{z-1}$; $(ROC)_X = (1,\infty)$

- Real $x[n] \Rightarrow \hat{X}(-w) = \left(\hat{X}(w)\right)^*$

  Even: $x[-n] = x[n] \Rightarrow$ even: $\hat{X}(-w) = \hat{X}(w)$

  Odd: $x[-n] = -x[n] \Rightarrow$ odd: $\hat{X}(-w) = -\hat{X}(w)$

  Real & even $x[n] \Rightarrow$ real & even $\hat{X}(w)$

  Real & odd $x[n] \rightarrow$ pure imaginary & odd $\hat{X}(w)$