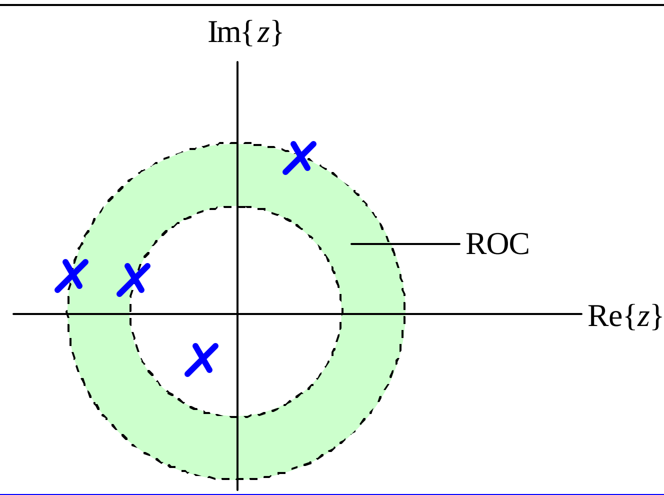
Math review

- $$\sum_{n=N_1}^{N_2} g^n = \frac{g^{N_1} - g^{N_2+1}}{1-g}$$
- $$\sum_{n=0}^{\infty} g^n = \begin{cases} \frac{1}{1-g} & if \quad |g| < 1 \\ diverges & if \quad |g| > 1 \end{cases}$$
- $$\sum_{n=0}^{\infty} n g^n = \begin{cases} \frac{g}{(1-g)^2} & if\ |g| < 1 \\ diverges\ if\ |g| > 1 \end{cases}$$
- $$\sin\left(n\frac{p}{2}\right) = \begin{cases} 0 & n \text{ even} \\ (-1)^{\frac{n-1}{2}} & n \text{ odd} \end{cases}$$
- $$\cos\left(n\frac{p}{2}\right) = \begin{cases} 0 & n \text{ odd} \\ (-1)^{\frac{n}{2}} & n \text{ even} \end{cases}$$

Z-transform

- $X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$; $0 \le R_a < |z| < R_b \le \infty$



$x[n]$	$X(z)$	|z| Î
Time-shift: $x[n-n_0]$	$z^{-n_0} X(z)$	same ROC

Proof $\sum_{n=-\infty}^{\infty} x[n-n_0] z^{-n} = \sum_{m=-\infty}^{\infty} x[m] z^{-(m+n_0)}$; $m = n - n_0$

$$= \left(\sum_{m=-\infty}^{\infty} x[m] z^{-m} \right) z^{-n_0}$$

Convolution: $x_1[n]*x_2[n]$	$X_1(z)\cdot X_2(z)$	ROC ⊃ $(ROC)_1 \cap (ROC)_2$

Proof $\sum_{n=-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty} x_1[k]\,x_2[n-k]\right)z^{-n} = \sum_{k=-\infty}^{\infty}\left(x_1[k]\left(\sum_{n=-\infty}^{\infty} x_2[n-k]\,z^{-n}\right)\right)$

$$= \sum_{k=-\infty}^{\infty}\left(x_1[k]\left(z^{-k}X_2(z)\right)\right)$$

$$= X_2(z)\sum_{k=-\infty}^{\infty}\left(x_1[k]\left(z^{-k}\right)\right)$$

$$= X_1(z)X_2(z)$$

$(n-1)x[n-1]$	$-\frac{d}{dz}X(z)$	same ROC

Proof $\frac{d}{dz}X(z) = \sum_{n=-\infty}^{\infty}(-n)x[n]z^{-n-1} = \sum_{m=-\infty}^{\infty} -(m-1)x[m-1]z^{-m} \quad ;m=n+1$

$$-\frac{d}{dz}X(z) = \sum_{m=-\infty}^{\infty}\left((m-1)x[m-1]\right)z^{-m}$$

$d[n]$	1	$(0,\infty)$

Proof $\sum_{n=-\infty}^{\infty} d[n]z^{-n} = \ldots + 0 + 1z^0 + 0 + \ldots = 1$

$d[n+n_0]$	z^{n_0}	$(0,\infty)$

Proof $\sum_{n=-\infty}^{\infty} d[n+n_0]z^{-n} = \ldots + 0 + 1z^{-(-n_0)} + 0 + \ldots = 1 = z^{n_0}$

Proof2 Use time-shift rule and $d[n] \overset{Z}{\rightleftharpoons} 1$.

$z_0^n u[n]$	$\frac{z}{z-z_0}$	$(\lvert z_0\rvert,\infty)$
$-z_0^n u[-n-1]$		$(0,\lvert z_0\rvert)$

Proof $\sum_{n=-\infty}^{\infty} z_0^n u[n]z^{-n} = \sum_{n=0}^{\infty} z_0^n z^{-n} = \sum_{n=0}^{\infty}\left(\frac{z_0}{z}\right)^n = \frac{1}{1-\frac{z_0}{z}} = \frac{z}{z-z_0}$

Need $\left|\frac{z_0}{z}\right| < 1$

Proof

$$\sum_{n=-\infty}^{\infty} -z_0^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{\infty} \left(\frac{z_0}{z}\right)^n u[-n-1]$$

$$= -\sum_{m=-\infty}^{\infty} \left(\frac{z_0}{z}\right)^{-(m+1)} u[m] \quad ; m = -n-1; n = -(m+1)$$

$$= -\sum_{m=0}^{\infty} \left(\frac{z}{z_0}\right)^{m+1} = -\frac{z}{z_0} \sum_{m=0}^{\infty} \left(\frac{z}{z_0}\right)^m = -\frac{z}{z_0} \frac{1}{1-\frac{z}{z_0}} = \frac{z}{z_0 - z}$$

Need $\left|\frac{z}{z_0}\right| < 1$

• $z_0^{n+n_0} u[n+n_0]$	$\frac{z^{n_0+1}}{z-z_0}$	$(\lvert z_0\rvert,\infty)$

Proof $z_0^n u[n] \overset{z}{\rightleftharpoons} \frac{z}{z-z_0}$; $ROC = (\lvert z_0\rvert, \infty)$

$x[n+n_0] \overset{z}{\rightleftharpoons} z^{n_0} X(z)$ same ROC

$z_0^{n-1} u[n-1]$	$\frac{1}{z-z_0}$	$(\lvert z_0\rvert,\infty)$
$-z_0^{n-1} u[-n]$		$(0,\lvert z_0\rvert)$

Proof Use time-shift rule: $x[n-n_0] \overset{z}{\rightleftharpoons} z^{-n_0} X(z)$

$x[n-1] \overset{z}{\rightleftharpoons} \frac{1}{z} X(z)$

From $z_0^n u[n] \overset{z}{\rightleftharpoons} \frac{z}{z-z_0}$,

$z_0^{n-1} u[n-1] \overset{z}{\rightleftharpoons} \frac{1}{z}\frac{z}{z-z_0}$

Proof Similarly, From $-z_0^n u[-n-1] \overset{z}{\rightleftharpoons} \frac{z}{z-z_0}$,

$-z_0^{n-1} u[-(n-1)-1] \overset{z}{\rightleftharpoons} \frac{1}{z}\frac{z}{z-z_0}$

$u[n]$	$\frac{z}{z-1}$	$(1,\infty)$
$-u[-n-1]$		$(0,1)$

Proof From z-transform of $z_0^n u[n]$ and $-z_0^n u[-n-1]$, letting $z_0 = 1$.

$n z_0^{n-1} u[n]$	$\frac{z}{(z-z_0)^2}$	$(\lvert z_0\rvert,\infty)$
$-n z_0^{n-1} u[-n-1]$		$(0,\lvert z_0\rvert)$

Proof $\frac{d}{dz_0} z_0^n u[n] \overset{z}{\rightleftharpoons} \frac{d}{dz_0} \frac{z}{z-z_0}$???

$$nz_0^{n-1}u[n] \overset{Z}{\rightleftharpoons} \frac{z}{(z-z_0)^2}$$

$\frac{n(n-1)}{2} z_0^{n-2}u[n]$	$\frac{z}{(z-z_0)^3}$	$(\lvert z_0\rvert,\infty)$
$-\frac{n(n-1)}{2} z_0^{n-2}u[-n-1]$		$(0,\lvert z_0\rvert)$
$\binom{n}{m} z_0^{n-m}u[n]$	$\frac{z}{(z-z_0)^{m+1}}$	$(\lvert z_0\rvert,\infty)$
$-\binom{n}{m} z_0^{n-m}u[-n-1]$		$(0,\lvert z_0\rvert)$
$z_0^{\lvert n\rvert}$	$\frac{z}{z-z_0} - \frac{z}{z-\frac{1}{z_0}}$	$\left(\lvert z_0\rvert, \frac{1}{\lvert z_0\rvert}\right)$

Proof $\quad z_0^{|n|} = z_0^n u[n] + z_0^{-n} u[-n-1]$

$$z_0^n u[n] \overset{Z}{\rightleftharpoons} \frac{z}{z-z_0} \quad ; ROC = (|z_0|,\infty)$$

$$-z_0^n u[-n-1] \overset{Z}{\rightleftharpoons} \frac{z}{z-z_0} \quad ; ROC = (0,|z_0|)$$

$$z_0^{-n} u[-n-1] \overset{Z}{\rightleftharpoons} -\frac{z}{z-\frac{1}{z_0}} \quad ; ROC = \left(0, \frac{1}{|z_0|}\right)$$

$$ROC = (|z_0|,\infty) \cap \left(0,\frac{1}{|z_0|}\right)$$

$\sum_{k=0}^{M-1} d[n-k]$	$\frac{1-z^{-M}}{1-z^{-1}}$	$(0,\infty)$

Proof $\quad X(z) = \sum_{k=0}^{M-1} z^{-k} = \frac{1-z^{-M}}{1-z^{-1}}$

Note: might suspect that there is a pole at $z = 1$, but there is not. Can see this from $X(z) = \sum_{k=0}^{M-1} z^{-k}$ or $\lim_{z\to 1} \frac{1-z^{-1}}{1-z^{-M}} = \lim_{x\to 1} \frac{1-x}{1-x^M} = \lim_{x\to 1} \frac{-1}{-Mx^{M-1}} = \frac{1}{M}$

- no pole of $X(z)$ lies in $(\text{ROC})_X$
- all finite edges of $(\text{ROC})_X$ must pass through one or more poles

- For $R_a < \hat{R} < R_b$, $x[n] = \frac{1}{2p} \int_0^{2p} X\left(\hat{R}e^{jq}\right) \hat{R}^n e^{jnq}\, dq$

Proof $\quad X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$

$$X\left(z=\hat{R}e^{jq}\right)=\sum_{m=-\infty}^{\infty}x[m]\hat{R}^{-m}e^{-jmq}$$

$$e^{jnq}X\left(\hat{R}e^{jq}\right)=\sum_{m=-\infty}^{\infty}x[m]\hat{R}^{-m}e^{-jmq}e^{jnq}=\sum_{m=-\infty}^{\infty}x[m]\hat{R}^{-m}e^{-j(m-n)q}$$

$$=\ldots+x[n-1]\,\hat{R}^{-(n-1)}e^{-j(-1)q}+x[n]\,\hat{R}^{-n}\,\cancel{e^{-j(0)q}}^{\,1}$$

$$+\,x[n+1]\,\hat{R}^{-(n+1)}e^{-j(1)q}+\ldots$$

$$\text{Since }\int_0^{2p}e^{-jkq}dq=\begin{cases}2p & k=0\\ \dfrac{1}{-jk}\cancel{\left(e^{-jk2p}-e^{-jk0}\right)}^{\,0} & k\neq 0\end{cases}=\begin{cases}2p & k=0\\ 0 & k\neq 0\end{cases}$$

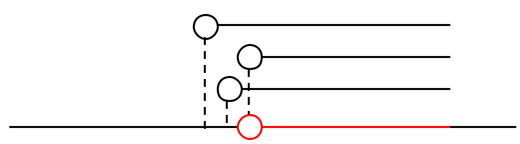
$$\int_0^{2p}e^{jnq}X\left(\hat{R}e^{jq}\right)dq=\ldots+0+\left(x[n]\hat{R}^{-n}\right)2p+0+\ldots=2p\,x[n]\,\hat{R}^{-n}$$

- Recipe for proper & rational *X(z)*:

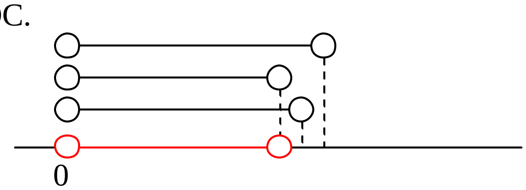
  Expand $\dfrac{X(z)}{z}$ in partial fractions

  - Inward-pole terms → "u[n]-part"
  - Outward-pole terms → "u[-n-1]-part"

  To see this, note that the ROC of a *u*[*n*] term will be in the form ($|z_0|,\infty$). Thus, to satisfy the ROC of all the *u*[*n*] terms, the ROC (the intersected result) will be of the form $\left(\max\left(|z_{in}|\right),\infty\right)$. So, all *u*[*n*] terms will have poles on the left-side of the combined ROC.



  On the other hand, the ROC of the *u*[-*n*-1] term will be in the form ($0,|z_0|$). Thus, to satisfy the ROC of all the *u*[-*n*-1] terms, the ROC (the intersected result) will be of the form $\left(0,\min\left(|z_{in}|\right)\right)$. So, all *u*[-*n*-1] terms will have poles on the right-side of the combined ROC.



  To have non-empty combined ROC between the ROC of all the *u*[*n*] terms and *u*[-*n*-1] terms, need to have all poles of *u*[*n*] terms on the left-side or inward-side, and all poles of *u*[-*n*-1] terms on the right-side or outward-side of the ROC.

- If *x*[*n*] is causal, then there are only the *u*[*n*] terms. Thus, if know that *x*[*n*] is causal, knowing the poles give the ROC = $\left(\max\left(|z_{in}|\right),\infty\right)$ or complex plane (just) outside all poles of *X*(*z*).

- If $\frac{X(z)}{z}$ is not strictly proper,
  - Write $\frac{X(z)}{z}$ = {polynomial in (z)} + {strictly proper}
  - Use $Kd(n+m_0) \overset{Z}{\rightleftharpoons} Kz^{m_0}$

Unilateral z-transform

- $x[n] \overset{Z_I}{\rightleftharpoons} X_I(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$

$x[n]u[n] \overset{Z}{\rightleftharpoons} X_I(z)$

- $X_I(z)$ tells nothing about $x[n]$ for $n < 0$
- $x[n]u[n]$ is causal $\Rightarrow$ ROC for $X_I(z)$ goes all the way out to ∞

- $x[n-1] \overset{Z_I}{\rightleftharpoons} z^{-1} X_I(z) + x[-1]$

Proof $\sum_{n=0}^{\infty} x[n-1] z^{-n} = \sum_{n=0}^{\infty} x[n-1] z^{-(n-1)} z^{-1} = z^{-1} \sum_{m=-1}^{\infty} x[m] z^{-(m)} \quad ; m = n-1$

$$= z^{-1} \sum_{m=0}^{\infty} x[m] z^{-(m)} + \cancel{z^{-1}} x[-1] \cancel{z^{1}}$$

- $x[n-2] \overset{Z_I}{\rightleftharpoons} z^{-2} X_I(z) + z^{-1} x[-1] + x[-2]$

Proof Let $y[n] = x[n-1] \Rightarrow Y_I(z) = z^{-1} X_I(z) + x[-1]$.

Then $y[n-1] \overset{Z_I}{\rightleftharpoons} z^{-1} Y_I(z) + y[-1] = z^{-1}\left(z^{-1} X_I(z) + x[-1]\right) + x[-2]$

- $d[n] \overset{Z_I}{\rightleftharpoons} 1$

Proof $d[n] = d[n]u[n]$

- Use for solving difference equations subject to appropriate initial conditions
  - Take z_I-transform of both sides
  - Solve for $Y_I(z)$; $ROC \Rightarrow$ complex plane outside all poles
  - Use recipe to get $y[n]u[n] \Rightarrow$ same as $y[n]$, $\forall n$ (because causal)