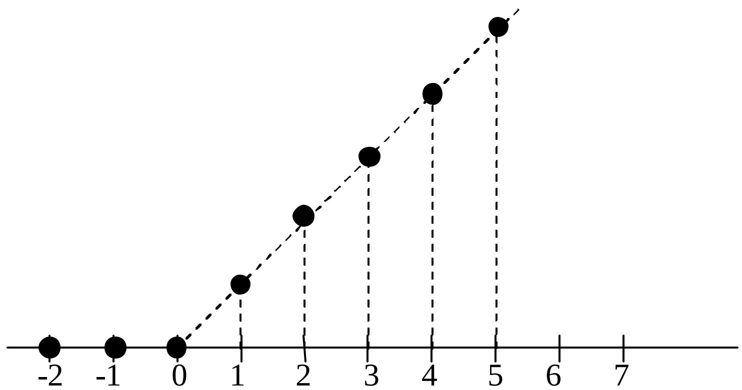
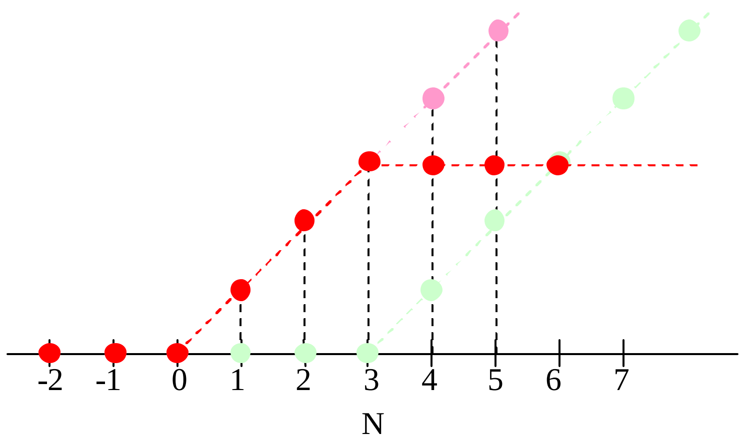
DISCRETE-TIME SIGNALS AND SYSTEMS

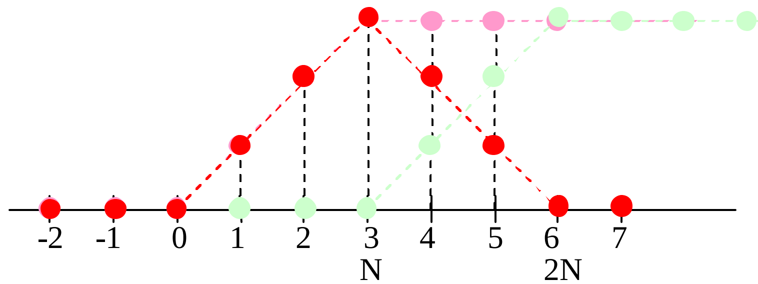
- Same $x[n]$ can have different-looking representations
  - Example
    - $u[n]-u[n-m]=\sum_{k=0}^{k=m-1} d[n-k]$
    - $n\cdot f[n]\cdot u[n]=n\cdot f[n]\cdot u[n-1]$
- Example
  - $x_1[n]=\begin{Bmatrix} 0 & n<0 \\ n & n\geq 0 \end{Bmatrix}=nu[n]$



  - $x_2[n]=\begin{Bmatrix} 0 & n<0 \\ n & 0\leq n<N \\ N & n\geq N \end{Bmatrix}=x_1[n]-x_1[n-N]$



- $x[n] = \begin{cases} 0 & n < 0 \\ n & 0 \le n < N \\ 2N - n & N \le n < 2N \\ 0 & n \ge 2N \end{cases} = x_2[n] - x_2[n - N]$



- $x[n] = \begin{cases} f[n] & \text{n even} \\ g[n] & \text{n odd} \end{cases} = f[n]\frac{1}{2}\left(1 + (-1)^n\right) + g[n]\frac{1}{2}\left(1 - (-1)^n\right)$

- $d[n]\, f[n] = f[0]$

  $d[n - n_0]\, f[n] = f[n_0]$

  $d[n] * f[n] = \sum_{k=-\infty}^{\infty} d[k]\, f[n-k] = f[n]$; identity element for convolution

  $d[n - n_0] * f[n] = \sum_{k=-\infty}^{\infty} d[k - n_0]\, f[n-k] = f[n - n_0]$

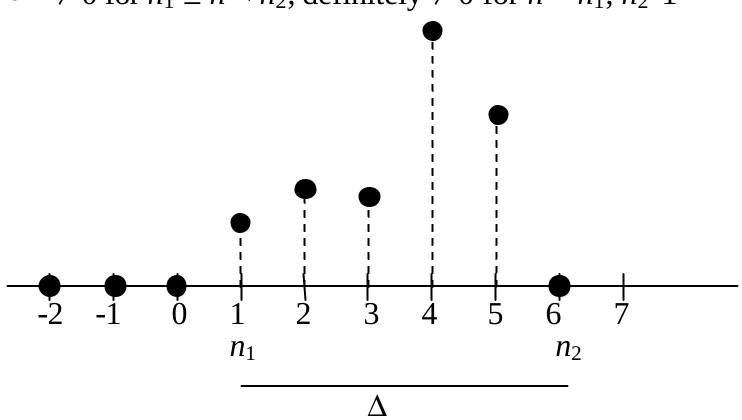
  $d[n] = \frac{\sin(np)}{np}$

- **Duration** D of $x[n] = n_2 - n_1$

  n_1 = largest integer satisfying $x[n<n_1] = 0$

  n_2 = smallest integer satisfying $x[n{}^{3}n_2] = 0$

  - $\neq 0$ for $n_1 \le n < n_2$, definitely $\neq 0$ for $n = n_1, n_2$-1



- $x[n]=\sum_{k=-\infty}^{\infty} a_k d[k]=\sum_{k=n_1}^{n_2-1} a_k d[k]$

- **Discrete-time convolution**

  $x_1[n]*x_2[n]=\sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k]=\sum_{k=-\infty}^{\infty} x_2[k]x_1[n-k]$; $\forall$n

  - Example:
    - $u[n]*(z[n]\cdot u[n])=\sum_{k=-\infty}^{\infty} z[k]u[k]u[n-k]=\left(\sum_{k=0}^{n} z[k]\right)u[n]$; $\forall$n
  - Duration of $x_1[n]*x_2[n]$ = $x_1[n]*x_2[n]=\Delta_1+\Delta_2-1$

    Proof Because $x_1[n]*x_2[n]=\sum_{k=-\infty}^{\infty} x_1[k]\,x_2[n-k]$, the sum is non-zero for n that satisfies $m_1\le n-k\le m_2-1$ where $n_1\le k\le n_2-1$.

    $\Rightarrow m_1+k\le n\le m_2-1+k$

    Therefore, the maximum value of n that could give non-zero result is $m_2-1+k_{\max}=m_2-1+n_2-1=m_2+n_2-2$. The minimum value of n that could give non-zero result is $m_1+k_{\min}=m_1+n_1$.

    Thus, duration: $m_1+n_1\le n\le m_2+n_2-2$ or $m_1+n_1\le n<m_2+n_2-1$.

    Duration = $(m_2+n_2-1)-(m_1+n_1)=(m_2-m_1)+(n_2-n_1)-1$

- Time invariance property of convolution:

  $x_1[n]*x_2[n]=y[n]\Rightarrow x_1[n]*x_2[n-n_0]=y[n-n_0]$

  Proof Let $x_3[n]=x_2[n-n_0]$, then

  $$x_1[n]*x_2[n-n_0]=x_1[n]*x_3[n]=\sum_{k=-\infty}^{\infty} x_1[k]\,x_3[n-k]$$

  $$=\sum_{k=-\infty}^{\infty} x_1[k]\,x_2[n-n_0-k]=\sum_{k=-\infty}^{\infty} x_1[k]x_2[(n-n_0)-k]$$

  Because $y[n]=\sum_{k=-\infty}^{\infty} x_1[k]\,x_2[n-k]$,

  $$\sum_{k=-\infty}^{\infty} x_1[k]x_2[(n-n_0)-k]=y[n-n_0]$$

  - $x_1[n]*x_2[n]=y[n]\Rightarrow x_1[n-n_1]*x_2[n-n_2]=y[n-n_1-n_2]$

    Proof 1

    Start with $x_1[n-n_1]*x_2[n]=y[n-n_1]$.

    Let $x_3[n]=x_1[n-n_1]$, and $y_1[n]=y[n-n_1]$.

So, the first equation becomes $x_3[n]*x_2[n]=y_1[n]$.

Then, use $x_3[n]*x_2[n-n_2]=y_1[n-n_2]$.

Thus, $x_1[n-n_1]*x_2[n-n_2]=y[n-n_1-n_2]$.

LTI and Z-transform

- $d[n] \xrightarrow{S}$ **Impulse response** $h[n] \underset{}{\overset{Z}{\rightleftharpoons}}$ **Transfer function** $H(z)$; $(ROC)_H$
- System is causal $\Leftrightarrow$ $h[n]$ is causal

  Proof $\quad y[n]=\sum_{k=-\infty}^{\infty} w[k]h[n-k]$

  $= \sum_{k=-\infty}^{n} w[k]h[n-k]$ for causal $h[n]$

  Thus, $y[n]$ is a function of $w[k]$ only for $k \le n$.

- $w[n] \xrightarrow{S_{LTI}} y[n] \overset{Z}{\rightleftharpoons} Y(z)=H(z)W(z)$; $(ROC)_Y = (ROC)_H \cap (ROC)_W$
- $z_0^n \xrightarrow{S_{LTI}} H(z_0)z_0^n$, $\forall n$, for $z_0 \in (ROC)_H$

  Proof $\quad z_0^n * h[n] = \sum_{k=-\infty}^{\infty} h[k]z_0^{n-k} = \left(\sum_{k=-\infty}^{\infty} h[k]z_0^{-k}\right) z_0^n = \left(H(z)\big|_{z=z_0}\right) z_0^n$

  - $1 \xrightarrow{S_{LTI}} H(1)$; for $1 \in (ROC)_H$
  - $a \xrightarrow{S_{LTI}} aH(1)$; for $a \in (ROC)_H$

- $u[n] \xrightarrow{S} y_s[n]$
- $h[n]=y_s[n]-y_s[n-1]$

  Proof $\quad u[n]-u[n-1] \xrightarrow{S} y_s[n]-y_s[n-1]$

  Because $d[n]=u[n]-u[n-1]$ and $d[n] \xrightarrow{S} h[n]$,

  $h[n]=y_s[n]-y_s[n-1]$

- $w[n] \xrightarrow{d[n]} w[n]$; therefore $\xrightarrow{d[n]} \equiv \longrightarrow$
- Stability of causal discrete-time LTI system

  BIBO stable $\leftrightarrow$

  - Bounded $w[n] \xrightarrow{S_{LTI}}$ well-defined and bounded $y[n]$
  - $h[n]$ is absolutely summable: $\sum_{n=0}^{\infty} |h[n]| < \infty$
    - Note: the sum starts @ n = 0 because $h[n]$ is causal
  - All poles of $H(z)$ lie in $|z| < 1$, for rational $H(z)$

≡ $(ROC)_H$ must include the unit circle

- LTI & **FIR (finite impulse response)** system
  - ⇔ $h_{FIR}[n]$ has finite duration: $n_1 \le n < n_2$ (need not be causal)

⇒

  - $H_{FIR}(z)$ has only one pole at $z = 0$

    Because $H_{FIR}(z) = \sum_{n=n_1}^{n_2-1} h[n]z^{-n}$

    always has ROC: $0 < |z| < \infty$
  - Every (causal) FIR system is stable
  - Causal FIR system ($n_1 \ge 0$) → tapped delay lines

Causal system governed by difference equations

- Assumption: system is causal ⇒ $h(t)$ is causal
- $\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{N} b_k w[n-k]$; try to make $a_0 = 1$
- Example

$$y[n]+a_1 y[n-1]+a_2 y[n-2]=b_0 w[n]+b_1 w[n-1]+b_2 w[n-2]$$

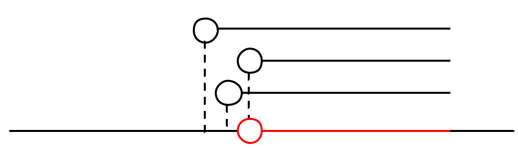
$$\overset{Z}{\rightleftharpoons} Y(z)+a_1 z^{-1}Y(z)+a_2 z^{-2}Y(z)=b_0 W(z)+b_1 z^{-1}W(z)+b_2 z^{-2}W(z)$$

$$H(z)=\frac{Y(z)}{W(z)}=\frac{b_0+b_1 z^{-1}+b_2 z^{-2}}{1+a_1 z^{-1}+a_2 z^{-2}}=\frac{b_0 z^2+b_1 z+b_2}{z^2+a_1 z+a_2} \text{ rational}$$

$(ROC)_H$ = complex plane (just) outside all poles of H(z)

$h(t)$ is causal ⇒ no $u[-n-1]$-part ⇒ all poles are inward

To see this, note that the ROC of a $u[n]$ term will be in the form $(|z_0|,\infty)$. Thus, to satisfy the ROC of all the $u[n]$ terms, the ROC (the intersected result) will be of the form $\left(\max\left(\left|z_{in}\right|\right),\infty\right)$. So, all $u[n]$ terms will have poles on the left-side of the combined ROC.



- Could implement any causal discrete-time LTI system with a proper rational H(z) using a difference equation.
- Other ways ⇒ introduce a pole-zero cancellation
- Example
  - M-fold sliding-windows averager:

$$w[n] \xrightarrow{S_{LTI}} y[n] = \frac{1}{M}\sum_{\ell=0}^{M-1} w[n-\ell] = \frac{1}{M}\sum_{k=n-(M-1)}^{n} w[k]$$

$$y[n] - y[n-1] = \frac{1}{M}\sum_{k=n-(M-1)}^{n} w[k] - \frac{1}{M}\sum_{k=n-(M-1)-1}^{n-1} w[k]$$

$$= \frac{1}{M}\left(w[n] - w[n-M]\right)$$

- From system definition, can easily see that $h[n] = \frac{1}{M}\sum_{k=0}^{M-1} d[n-k] \Rightarrow$ FIR

  This gives $H(z) = \frac{1}{M}\sum_{k=0}^{M-1} z^{-k} = \frac{1}{M}\frac{1-z^{-M}}{1-z^{-1}}$; $0 < |z| < \infty$

- From $y[n] - y[n-1] = \frac{1}{M}\left(w[n] - w[n-M]\right)$,

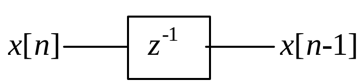
  $Y(z) - z^{-1}Y(z) = \frac{1}{M}\left(W(z) - z^{-M}W(z)\right)$

  $\Rightarrow H(z) = \frac{Y(z)}{W(z)} = \frac{1}{M}\frac{1-z^{-1}}{1-z^{-M}}$; $\forall$z

  Note: no pole @ z = 1. $\lim_{z\to 1}\frac{1}{M}\frac{1-z^{-1}}{1-z^{-M}} = \frac{1}{M}\lim_{x\to 1}\frac{1-x}{1-x^{M}} = \frac{1}{M}\lim_{x\to 1}\frac{-1}{-Mx^{M-1}} = \frac{1}{M^2}$

Block Diagrams

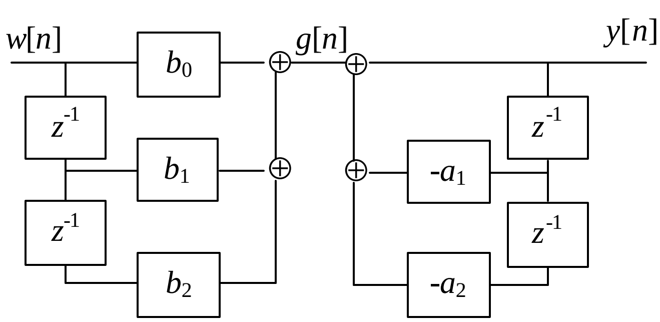
- $h(t) = d[n-1] \overset{Z}{\rightleftarrows} z^{-1}$

- Direct Form I

  $g[n] = b_0 w[n] + b_1 w[n-1] + b_2 w[n-2] = y[n] + a_1 y[n-1] + a_2 y[n-2]$

  $y[n] = g[n] - a_1 y[n-1] - a_2 y[n-2]$

- Direct Form II

  Controllable canonical realization

$$Y(z)=\frac{b_0+b_1z^{-1}+b_2z^{-2}}{1+a_1z^{-1}+a_2z^{-2}}W(z)=\left(b_0+b_1z^{-1}+b_2z^{-2}\right)\underbrace{\left(\frac{W(z)}{1+a_1z^{-1}+a_2z^{-2}}\right)}_{Q(z)}$$

$$Q(z)=\frac{W(z)}{1+a_1z^{-1}+a_2z^{-2}}$$

$$Q(z)+a_1z^{-1}Q(z)+a_2z^{-2}Q(z)=W(z)$$

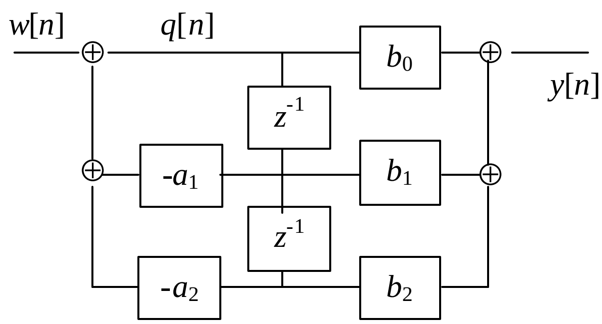
$$q[n]+a_1q[n-1]+a_2q[n-2]=w[n]$$

$$q[n]=w[n]-a_1q[n-1]-a_2q[n-2]$$

$$Y(z)=\left(b_0+b_1z^{-1}+b_2z^{-2}\right)Q(z)$$

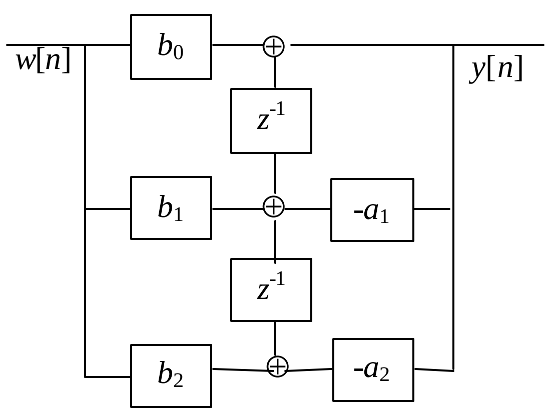
$$Y(z)=b_0Q(z)+b_1z^{-1}Q(z)+b_2z^{-2}Q(z)$$

$$y[n]=b_0q[n]+b_1q[n-1]+b_2q[n-2]$$



- Transposed Direct From II
  Observable canonical realization

$$y[n]=b_0w[n]+\left(b_1w[n-1]-a_1y[n-1]\right)+\left(b_2w[n-2]-a_2y[n-2]\right)$$



Discretization of continuous-time system

- Replace (t) → (nT) → [n]
- **Impulse equivalence/invariance**

$h_d[n] = Th(nT)\ ,\ \forall n$

$$h(t) = \sum e^{s_0 t} u(t)\,;\ H(s) = \sum \frac{1}{s - s_0}\,;\ (ROC)_H \Rightarrow \mathrm{Re}\{s\} \in \left(\left(\mathrm{Re}\{s_0\} \right)_{\max}, \infty \right)$$

$$h_d[n] = Th(nT) = \sum T e^{s_0 nT} u(nT) = \sum T \left(e^{s_0 T} \right)^n u[n]$$

$$H_d(z) = \sum T \frac{z}{z - e^{s_0 T}}\,;\ (ROC)_H \Rightarrow |z| \in \left({e^{s_0 T}}_{\max}, \infty \right)$$

- Perfect for band-limited $w(t)$ and $h(t)$, and T is shorter than Nyquist intervals of $w(t)$ and $h(t)$
- $w_d[n] = w(nT) \xrightarrow{S_{LTI}} y_d[n] = y(nT)$
- $h_d[n] = \left(e^{aT}\right)^n u[n] \ \Rightarrow\ h(t) = \frac{1}{T} e^{at} u(t) + f(t) u(t)$ where $f(nT) = 0$
- Pole $H(s)$ @ $s_0 \Rightarrow$ pole $H_d(z)$ @ $e^{s_0 T}$

- **Back-difference Approximation**: $Dy(t) \approx \frac{1}{T}\left(y(nT) - y((n-1)T) \right)$
  - $Dy(t) + ay(t) = bw(t) \ \Rightarrow\ H(s) = \dfrac{b}{s - (-a)}$

$$\frac{1}{T}\left(Ty[n] - Ty[n-1] \right) + aTy[n] = Tbw[n]$$

$$(1 + aT) y[n] - y[n-1] = Tbw[n] \ \Rightarrow\ (1 + aT) Y(z) - z^{-1} Y(z) = TbW(z)$$

$$H(z) = \frac{Y(z)}{W(z)} = \frac{Tb}{(1 + aT) - z^{-1}} = \frac{\dfrac{Tb}{1 + Ta}}{1 - \dfrac{z^{-1}}{1 + Ta}} = \frac{\dfrac{Tb}{1 + Ta} z}{z - \dfrac{1}{1 - (-a)T}}$$

  - $H_d(z) = H\left(s = \dfrac{1 - z^{-1}}{T} \right)$

$$\frac{T\cancel{b}}{(1 + aT) - z^{-1}} = \frac{\cancel{b}}{s - (-a)} \Rightarrow Ts + \cancel{Ta} = 1 + \cancel{aT} - z^{-1} \ \Rightarrow\ s = \frac{1 - z^{-1}}{T}$$

  - Pole $H(s)$ @ $s_0 \Rightarrow$ pole $H_d(z)$ @ $\dfrac{1}{1 - s_0 T}$

Forward-difference Approximation: $Dy(t) \approx \frac{1}{T}\left(y((n+1)T) - y(nT) \right)$

- $Dy(t) + ay(t) = bw(t) \ \Rightarrow\ H(s) = \dfrac{b}{s - (-a)}$

$$\frac{1}{\cancel{T}}\left(\cancel{T}y[n+1]-\cancel{T}y[n]\right)+aTy[n]=Tbw[n]$$

$$y[n+1]+(Ta-1)y[n]=Tbw[n]$$

$$zY(z)+(Ta-1)Y(z)=TbW(z)$$

$$H(z)=\frac{Y(z)}{W(z)}=\frac{Tb}{z-(1-Ta)}$$

- $H_d(z)=H\left(s=\dfrac{z-1}{T}\right)$

$$\frac{\cancel{b}}{s-(-a)}=\frac{T\cancel{b}}{z-(1-Ta)} \Rightarrow z-1+\cancel{Ta}=Ts+\cancel{Ta} \Rightarrow s=\frac{z-1}{T}$$

- The discretization is stable $\leftrightarrow$ $|1-aT|<1$ & $T>0$

- **Bilinear Transformation**:

$$y(t)=\int w(t)dt \Rightarrow y(nT)-y((n-1)T)\approx\frac{T}{2}\left(w(nT)+w((n-1)T)\right)$$

$$y(nT)-y((n-1)T)\approx\frac{1}{2}\left(w(nT)+w((n-1)T)\right)$$

$$y[n]-y[n-1]=\frac{T}{2}w[n]-\frac{T}{2}w[n-1]$$

  - $H_d(z)=H\left(s=\dfrac{2}{T}\left(\dfrac{z-1}{z+1}\right)\right)$
  - Pole $H(s)$ @ s_0 $\Rightarrow$ pole $H_d(z)$ @ $\dfrac{2+Ts_0}{2-Ts_0}$

- To find the new pole from the replacement: $H_d(z)=H(s=f(z))$

  $H(s)$: has pole when $s=s_0$

  $\therefore$ $H(z)$ has pole when $s_0=f(z_0)$ $\Rightarrow$ solve for z_0

- **Stability** : If Re$\{s_0\}<0$ for every pole of H(s),

  Then $|z_0|<1$ for every pole z_0 of $H_d(z)$ by any method

LTI and DTFT

• $d[n]\xrightarrow{S}$ **Impulse response** $h[n]$ $\overset{DTFT}{\rightleftharpoons}$ **Frequency response** $\hat{H}(w)$
• If $\hat{W}(w)$ exists, $w[n]\xrightarrow{S_{LTI}}y[n]=h[n]*w[n]\overset{DTFT}{\rightleftharpoons}\hat{Y}(w)=\hat{H}(w)\hat{W}(w)$
• $e^{jnw_0}\xrightarrow{S_{LTI}}\left(\hat{H}(w)\big\vert_{w=w_0}\right)e^{jnw_0}$

Proof $h[n]*e^{jnw_0} = \sum_{k=-\infty}^{\infty} h[k]e^{j(n-k)w_0} = e^{jnw_0}\sum_{k=-\infty}^{\infty} h[k]e^{jkw_0} = \left(\hat{H}(w)\big|_{w=w_0}\right)e^{jnw_0}$

- Example
  - M-fold sliding-window averager: $h[n] = \frac{1}{M}\sum_{k=0}^{M-1} d[n-k]$

$$\hat{H}(w) = \frac{1}{M}e^{-j\frac{M-1}{2}w}\left(\frac{\sin\left(\frac{M}{2}w\right)}{\sin\left(\frac{1}{2}w\right)}\right)$$

because $\sum_{k=0}^{M-1} d[n-k] \xrightleftharpoons{DTFT} e^{-j\frac{M-1}{2}w}\left(\frac{\sin\left(\frac{M}{2}w\right)}{\sin\left(\frac{1}{2}w\right)}\right)$