Laplace Transforms

- 2-sided or bilateral Laplace transform

$$x(t) \overset{\mathcal{L}}{\rightleftharpoons} X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$ROC: s_a < \mathrm{Re}\{s\} < s_b \Rightarrow$ region of convergence $\Rightarrow$ s-values for which the integral converges

ROC $\Rightarrow$ vertical strip in the complex plane

- $e^{s_0 t}$ are not Laplace Transformable
- If *x(t)* is right-sided ($x(t<t_0) = 0$),

  then $s_b = \infty \Rightarrow$ ROC $\Rightarrow$ $s_a < \mathrm{Re}\{s\} < \infty$

  If *x(t)* is left-sided ($x(t>t_0) = 0$),

  then $s_a = \infty \Rightarrow$ ROC $\Rightarrow$ $-\infty < \mathrm{Re}\{s\} < s_b$

  If *x(t)* is time-limited,

  then ROC $\Rightarrow$ $-\infty < \mathrm{Re}\{s\} < \infty$

x(t)	*X(s)*	ROC $\Rightarrow$ Re{s} Î
$c_1 x_1(t) + c_2 x_2(t)$	$c_1 X_1(s) + c_2 X_2(s)$	(usually) (ROC)$_1 \cap$ (ROC)$_2$

Linearity
Need (ROC)$_1 \cap$ (ROC)$_2 \neq \phi$

$x(t - t_0)$	$e^{-st_0} X(s)$	same ROC

Time-shift rule

$\frac{d}{dt} x(t)$	$sX(s)$	(usually) same or at least, contain (ROC)$_X$

Time-derivative rule

$x_1(t) * x_2(t)$	$X_1(s) \cdot X_2(s)$	(ROC)$_1 \cap$ (ROC)$_2$

Convolution rule
Need (ROC)$_1 \cap$ (ROC)$_2 \neq \phi$

$e^{s_0 t} u(t)$	$\frac{1}{s - s_0}$	(Re{s_0},∞)
$-e^{s_0 t} u(-t)$		($-\infty$,Re{s_0})

Proof $$\int_{-\infty}^{\infty} e^{s_0 t} u(t) e^{-st} dt = \int_0^{\infty} e^{s_0 t} e^{-st} dt = \int_0^{\infty} e^{-(s-s_0)t} dt = \frac{1}{s - s_0}$$

Need $\mathrm{Re}\{s - s_0\} > 0$ or $\mathrm{Re}\{s\} > \mathrm{Re}\{s_0\}$

Proof $$\int_{-\infty}^{\infty} -e^{s_0 t}u(-t)e^{-st}dt = -\int_{-\infty}^{0} e^{s_0 t}e^{-st}dt = -\int_{-\infty}^{0} e^{(s_0 - s)t}dt = -\frac{1}{s_0 - s} = \frac{1}{s - s_0}$$

Need $\mathrm{Re}\{s_0 - s\} > 0$ or $\mathrm{Re}\{s\} < \mathrm{Re}\{s_0\}$

$te^{s_0 t}u(t)$	$\dfrac{1}{(s-s_0)^2}$	(Re{s_0},∞)
$-te^{s_0 t}u(-t)$		(-∞,Re{s_0})
$\dfrac{t^k}{k!}e^{s_0 t}u(t)$	$\dfrac{1}{(s-s_0)^{k+1}}$	(Re{s_0},∞)
$-\dfrac{t^k}{k!}e^{s_0 t}u(-t)$		(-∞,Re{s_0})
u(t)	$\dfrac{1}{s}$	(0,∞)
-u(-t)		(-∞,0)
d(t)	1	(-∞,∞)
d(t-T)	e^{-sT}	(-∞,∞)
$P_T(t)$	$\dfrac{e^{sT} - e^{-sT}}{s}$	(-∞,∞)
$u(t)\cos(w_o t)$	$\dfrac{s}{s^2 - w_0^2}$	(0,∞)
$e^{-s_0\lvert t\rvert} = e^{-s_0 t}u(t) + e^{s_0 t}u(-t)$	$\dfrac{-2s_0}{s^2 - s_0^2}$	(-Re{s_0},Re{s_0})

- $\mathscr{L}$-transform and $\Im$-transform
  - If imaginary axis lies inside $(\mathrm{ROC})_X$ ($\sigma_a < 0 < \sigma_b$), then $\hat{X}(w)$ exists and $= X(s = jw)$
  - If the imaginary axis is bounded away from $(\mathrm{ROC})_X$, then, *x(t)* has no $\Im$-transform
  - If the imaginary axis is an "edge" of $(\mathrm{ROC})_X$, then, x(t) may or may not have a $\Im$-transform
    - won't in general have $\hat{X}(w) = X(s = jw)$
    - $\hat{X}(w)$ will usually contain *d*

Recovering *x(t)* from *X(s)* , $(\mathrm{ROC})_X$

- Pick $\hat{s}$ so that line s = $\hat{s}$ + jω is inside $(\mathrm{ROC})_X$

$$x(t) = \frac{1}{2p} \int_{-\infty}^{\infty} X(\hat{s} + jw) e^{(\hat{s}+jw)t} dw$$

- Given $X(s) = \dfrac{p(s)}{q(s)}$

  roots of $p(s) \Rightarrow$ **zero**s of $X(s)$

  roots of $q(s) \Rightarrow$ **pole**s of $X(s)$

  - No poles of $X(s)$ lies in $(\text{ROC})_X$
  - Finite edge(s) of $(\text{ROC})_X$ pass through one or more poles

- Partial fraction

  - $$X(s) = \frac{r_1(s)}{r_2(s)} \frac{1}{s - s_0}\ ;\ r_1(s_0), r_2(s_0) \neq 0$$

    $$X(s) = \frac{K}{s - s_0} + r_3(s) \text{ where } K = \lim_{s \to s_0} \left((s - s_0) X(s) \right)$$

    Proof $$\frac{r_1(s)}{r_2(s)} \frac{1}{s - s_0} = \frac{K}{s - s_0} + r_3(s) \Rightarrow \frac{r_1(s)}{r_2(s)} = K + (s - s_0) r_3(s)$$

    $$\frac{r_1(s_0)}{r_2(s_0)} = K + 0 = K \Rightarrow K = \frac{r_1(s_0)}{r_2(s_0)} = \lim_{s \to s_0} \left((s - s_0) X(s) \right)$$

  - $$X(s) = \frac{r_1(s)}{r_2(s)} \frac{1}{(s - s_0)^j}\ ;\ r_1(s_0), r_2(s_0) \neq 0$$

    $$X(s) = \sum_{m=1}^{j} \frac{K_m}{(s - s_0)^m} + r_3(s) \text{ where}$$

    $$K_j = \lim_{s \to s_0} \left((s - s_0)^j X(s) \right)$$

    $$K_m = \frac{1}{(j-m)!} \lim_{s \to s_0} \left(\frac{d^{(j-m)}}{ds^{(j-m)}} \left((s - s_0)^j X(s) \right) \right)$$

    Proof $$\frac{r_1(s)}{r_2(s)} \frac{1}{(s - s_0)^j} = \sum_{m=1}^{j} \frac{K_m}{(s - s_0)^m} + r_3(s)$$

    $$\Rightarrow \frac{r_1(s)}{r_2(s)} = \sum_{m=1}^{j} K_m (s - s_0)^{j-m} + (s - s_0)^j r_3(s)$$

    $$= \sum_{m=1}^{j-1} K_m (s - s_0)^{j-m} + K_j \cancelto{1}{(s - s_0)^0} + (s - s_0)^j r_3(s)$$

    $$\frac{r_1(s_0)}{r_2(s_0)} = K_j$$

$$\frac{d^n}{ds^n}\frac{r_1(s)}{r_2(s)} = \sum_{m=1}^{j} K_m \frac{d^n}{ds^n}(s-s_0)^{j-m} + \frac{d^n}{ds^n}(s-s_0)^j r_3(s) \quad ; n < j - m$$

Note that $\frac{d^n}{ds^n}(s-s_0)^k = \begin{cases} 0 \text{ when } n > k \\ k! \text{ when } n = k \\ \frac{k!}{(k-n)!}(s-s_0)^{k-n} \text{ when } n < k \end{cases}$.

So, $\left.\left(\frac{d^n}{ds^n}(s-s_0)^k\right)\right|_{s=s_0} = \begin{cases} k! \text{ when } n = k \\ 0 \text{ otherwise} \end{cases}$.

Similarly, $\left.\left(\frac{d^n}{ds^n}(s-s_0)^{j-m}\right)\right|_{s=s_0} = \begin{cases} (j-m)! \text{ when } n = j-m \\ 0 \text{ otherwise} \end{cases}$

Thus, $\left.\left(\frac{d^n}{ds^n}\frac{r_1(s)}{r_2(s)}\right)\right|_{s=s_0} = (\ldots + 0 + K_{j-n}(n!) + 0 + \ldots) + 0 = K_{j-n}(n!)$

And $K_{j-n} = \frac{1}{n!}\left.\left(\frac{d^n}{ds^n}\frac{r_1(s)}{r_2(s)}\right)\right|_{s=s_0} \Rightarrow K_m = \frac{1}{(j-m)!}\left.\left(\frac{d^{(j-m)}}{ds^{(j-m)}}\frac{r_1(s)}{r_2(s)}\right)\right|_{s=s_0}$

- $X(s) = \frac{p(s)}{\prod_i (s-s_i)} = \sum_i \frac{K_i}{s-s_i}$ where $K_i = \lim_{s \to s_i}((s-s_i)X(s))$
- $\frac{1}{s^2+a^2} = \frac{1}{2ia}\left(\frac{1}{s-ia} - \frac{1}{s+ia}\right)$

  $\frac{1}{s^2-a^2} = \frac{1}{2a}\left(\frac{1}{s-a} - \frac{1}{s+a}\right)$

  $\frac{s}{s^2-a^2} = \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right)$

  $\frac{s}{s^2+a^2} = \frac{1}{2}\left(\frac{1}{s-ia} + \frac{1}{s+ia}\right)$
- $\frac{s}{(s-a)^2} = \frac{1}{s-a} + \frac{a}{(s-a)^2}$

  $\frac{1}{(s-a)(s-b)} = \frac{\frac{1}{b-a}}{s-a} - \frac{\frac{1}{b-a}}{s-b}$

$$\frac{s}{(s-a)(s-b)} = \frac{\frac{a}{a-b}}{s-a} - \frac{\frac{b}{a-b}}{s-b}$$

- If *X(s)* is a strictly proper rational function of *s*
  - Graph $(\text{ROC})_X$ and mark the poles of *X(s)*
  - Expand X(s) = $\frac{p(s)}{q(s)}$ (degree p < q) in partial fractions
  - use prototypes
    - Leftward poles → the *u(t)*-part of *x(t)*
    - Rightward poles → the *u(-t)*-part of x(t)

- If know $z(t) \overset{\mathcal{L}}{\rightleftharpoons} \frac{p(s)}{q(s)}$, then $z(t-T) \overset{\mathcal{L}}{\rightleftharpoons} e^{-sT}\frac{p(s)}{q(s)}$

  Proof From $x(t-T) \overset{\mathcal{L}}{\rightleftharpoons} e^{-sT}X(s)$
- If degree *p* = degree *q*,

  write $Y(s) = K_0 + \frac{p_1(s)}{q(s)}$. Use $K_0 d(t) \overset{\mathcal{L}}{\rightleftharpoons} K_0$
- For *x(t)* to be **causal** and have *X(s)* as "the formula part" of its $\mathcal{L}$-transform,

  need all poles of *X(s)* to be to the left of $(\text{ROC})_X$ ⇒ have only u(t)-terms, no u(-t)-term
  - $x(t) = \sum_\ell k_\ell e^{s_\ell t}u(t) \overset{\mathcal{L}}{\rightleftharpoons} X(s) = \sum_\ell \frac{k_\ell}{s-s_\ell}$; $\text{Re}\{s\} > \max_\ell \text{Re}\{s_\ell\}$

    $(\text{ROC})_X = \bigcap_\ell (\text{Re}\{s_\ell\}, \infty) = \left(\max_\ell \text{Re}\{s_\ell\}, \infty\right)$
- For 1) rational H(s) 2) causal system,

  $(\text{ROC})_H$ = the part of complex plane to the right of all poles of H(s)

One-sided Laplace transform

- $x(t) \overset{\mathcal{L}}{\rightleftharpoons} X_I(s) = \int_0^\infty x(t)u(t)e^{-st}dt$
  - $\sigma_b \to +\infty$

- $Dx(t) \overset{\mathcal{L}_I}{\rightleftharpoons} X_I(s) - x(0^-)$

  $$D^2x(t) \overset{\mathcal{L}_I}{\rightleftharpoons} s\left(X_I(s) - x(0^-)\right) - Dx(0^-)$$

  $$\overset{\mathcal{L}_I}{\rightleftharpoons} s^2X_I(s) - s\cdot x(0^-) - Dx(0^-)$$

	$D^3x(t) \overset{\mathcal{L}_I}{\rightleftharpoons} s\left(s^2X_I(s) - s\cdot x(0^-) - Dx(0^-)\right) - D^2x(0^-)$
•	To convert $Y_I(s)$ back to $y(t)$ $(ROC)_{Y_I} \Rightarrow$ region to the right of all poles of $Y_I(s)$ Follow the same recipe as the inverse Laplace transform $\Rightarrow$ get $y(t)u(t) \Rightarrow$ cancel $u(t)$ $\Rightarrow$ get $y(t)$ for $t \geq 0$