Math Review

- Euler's formula: $e^{jw_0 t} = \cos(w_0 t) + j\sin(w_0 t)$

  $\cos(A) = \mathrm{Re}\left(e^{jA}\right) = \frac{1}{2}\left(e^{jA} + e^{-jA}\right)$

  $\sin(A) = \mathrm{Im}\left(e^{jA}\right) = \mathrm{Re}\left(-je^{jA}\right) = \mathrm{Re}\left(-\frac{1}{j}e^{jA}\right) = \frac{1}{2j}\left(e^{jA} - e^{-jA}\right)$

- $e^{jnA} = e^{jn(A+2kp)}$; $n \in$ Integer

- $\vec{s}_0 = s_0 + jw_0 = \sqrt{s_0^{\,2} + w_0^{\,2}}\, e^{j\tan^{-1}\left(\frac{w_0}{s_0}\right)} = \left|\vec{s}_0\right| e^{jf_0}$

- $\vec{A}^t = (a + jb)^t = \left|\vec{A}\right|^t e^{jf_A t} = e^{t\ln\left|\vec{A}\right| + jf_A t}$

  $A^t = \left(e^{\ln A}\right)^t = e^{t\ln A}$

- $\left|e^{jA}\right|^2 = e^{jA} \cdot \overline{e^{jA}} = e^{jA} \cdot e^{-jA} = 1$

- $e^{jAt} + e^{jBt} = e^{j\frac{A+B}{2}t}\left(e^{j\frac{A-B}{2}} + e^{-j\frac{A-B}{2}}\right) = 2e^{j\frac{A+B}{2}t}\cos\left(\frac{A-B}{2}\right)$

- $e^{jAt} - e^{jBt} = e^{j\frac{A+B}{2}t}\left(e^{j\frac{A-B}{2}} - e^{-j\frac{A-B}{2}}\right) = 2je^{j\frac{A+B}{2}t}\sin\left(\frac{A-B}{2}\right)$

- $\frac{e^{jAt} - e^{jBt}}{e^{jCt} - e^{jDt}} = e^{j\frac{(A+B)-(C+D)}{2}t}\,\frac{\sin\left(\frac{A-B}{2}\right)}{\sin\left(\frac{C-D}{2}\right)}$

  Proof $e^{jAt} - e^{jBt} = 2je^{j\frac{A+B}{2}t}\sin\left(\frac{A-B}{2}\right)$

  $e^{jCt} - e^{jDt} = 2je^{j\frac{C+D}{2}t}\sin\left(\frac{C-D}{2}\right)$

  $$\frac{e^{jAt} - e^{jBt}}{e^{jCt} - e^{jDt}} = \frac{\cancel{2j}\,e^{j\frac{A+B}{2}t}\sin\left(\frac{A-B}{2}\right)}{\cancel{2j}\,e^{j\frac{C+D}{2}t}\sin\left(\frac{C-D}{2}\right)} = e^{j\frac{(A+B)-(C+D)}{2}t}\,\frac{\sin\left(\frac{A-B}{2}\right)}{\sin\left(\frac{C-D}{2}\right)}$$

- $\vec{C}e^{\vec{s}_0 t} = \left|\vec{C}\right| e^{jq_C} e^{(s_0 + jw_0)t} = \left|\vec{C}\right| e^{s_0 t} e^{j(w_0 t + q_C)}$

  $= \left(\left|\vec{C}\right| e^{s_0 t}\cos(w_0 t + q_C)\right) + j\left(\left|\vec{C}\right| e^{s_0 t}\sin(w_0 t + q_C)\right)$

- $A_1 e^{s_0 t}\cos\left(w_0 t + f_1\right) + A_2 e^{s_0 t}\sin\left(w_0 t + f_2\right)$

  $= \mathrm{Re}\left(A_1 e^{s_0 t} e^{j(w_0 t + f_1)} - jA_2 e^{s_0 t} e^{j(w_0 t + f_1)}\right)$

  $= \mathrm{Re}\left(\left(A_1 e^{jf_{11}} - jA_2 e^{jf_{12}}\right) e^{(s_0 + jw_0)t}\right)$

  $= \mathrm{Re}\left(\left(A_1\left(\cos f_1 + j\sin f_1\right) - jA_2\left(\cos f_2 + j\sin f_2\right)\right) e^{(s_0 + jw_0)t}\right)$

  $= \mathrm{Re}\left(\left(\left(A_1\cos f_1 + A_2\sin f_2\right) + j\left(A_1\sin f_1 - A_2\cos f_2\right)\right) e^{(s_0 + jw_0)t}\right)$

  - $A\cos w_0 t + B\sin w_0 t = \mathrm{Re}\left(Ae^{jw_0 t}\right) + \mathrm{Re}\left(-jBe^{jw_0 t}\right) = \mathrm{Re}\left(\left(A - jB\right)e^{jw_0 t}\right)$

    $$= \mathrm{Re}\left(\sqrt{A^2 + B^2}\, e^{-j\tan^{-1}\frac{B}{A}} e^{jw_0 t}\right) = \sqrt{A^2 + B^2}\cos\left(w_0 t - \tan^{-1}\frac{B}{A}\right)$$

  - $A_1\cos\left(w_0 t + f_1\right) + A_2\sin\left(w_0 t + f_2\right)$

    $= \mathrm{Re}\left(A_1 e^{j(w_0 t + f_1)} - jA_2 e^{j(w_0 t + f_1)}\right) = \mathrm{Re}\left(\left(A_1 e^{jf_{11}} - jA_2 e^{jf_{12}}\right)e^{jw_0 t}\right)$

    $= \mathrm{Re}\left(\left(A_1\left(\cos f_1 + j\sin f_1\right) - jA_2\left(\cos f_2 + j\sin f_2\right)\right) e^{jw_0 t}\right)$

    $= \mathrm{Re}\left(\left(\left(A_1\cos f_1 + A_2\sin f_2\right) + j\left(A_1\sin f_1 - A_2\cos f_2\right)\right)\right)$

    $= \mathrm{Re}\left(\left(C + jD\right)e^{jw_0 t}\right)$

- **integrable**

  strong sense: *f(t)* is absolutely integrable $\leftrightarrow \lim_{\substack{T_1\to\infty \\ T_2\to\infty \\ \text{independently}}} \int_{-T_1}^{T_2} \left|f(t)\right| dt$ exists

  weaken sense: Cauchy Principle Value of $\int_{-\infty}^{\infty} f(t)dt = \lim_{T\to\infty}\int_{-T}^{T}\left|f(t)\right|dt$

- f(t) is **square integrable** $\leftrightarrow \lim_{\substack{T_1\to\infty \\ T_2\to\infty \\ \text{independently}}} \int_{-T_1}^{T_2}\left|f(t)\right|^2 dt$ exists

- $$\frac{1}{M}\sum_{\ell=0}^{M-1} e^{j2p\ell\frac{n}{M}} = \begin{cases} 1;\ \text{if } \frac{n}{M}\in I \\ 0;\ \text{if } \frac{n}{M}\notin I \end{cases}$$

  $$\sum_{\ell=0}^{M-1} e^{j2p\ell\frac{n}{M}} = \sum_{\ell=0}^{M-1}\left(e^{j2p\frac{n}{M}}\right)^{\ell} = \frac{1 - e^{j2p\frac{n}{M}M}}{1 - e^{j2p\frac{n}{M}}} = \frac{1 - e^{j2pn}}{1 - e^{j2p\frac{n}{M}}}$$

  $$= 0 \text{ if } \frac{n}{M}\notin I \text{ since } e^{j2p\frac{n}{M}} \neq 1$$

$$\text{If } \frac{n}{M} \in I, \; \sum_{\ell=0}^{M-1} e^{j2p\ell\frac{n}{M}} = \frac{1-\left(e^{j2p}\right)^{n}}{1-\left(e^{j2p}\right)^{\frac{n}{M}}}$$

$$= \frac{1-\left(e^{j2p}\right)^{n}}{1-\left(e^{j2p}\right)^{\frac{n}{M}}} = \lim_{x\to 1} \frac{1-x^{n}}{1-x^{\frac{n}{M}}} = \lim_{x\to 1} \frac{-nx^{n-1}}{-\frac{n}{M}x^{\frac{n}{M}-1}} = M$$

Signal

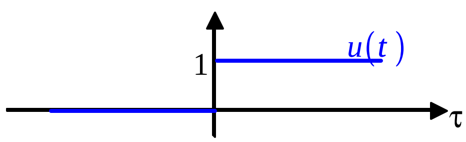
- Continuous-time signal: x(t)
- Discrete-time signal: x[n]

Continuous-time signal

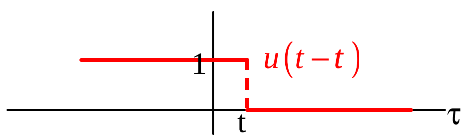
Examples

- **Unit step** $u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$

  - $u(t)$



  - $u(t-t)$



  - $y(t) = \begin{cases} f(t) & ; t \geq t_0 \\ g(t) & ; t < t_0 \end{cases} = f(t)u(t-t_0) + g(t)u(t_0-t), \forall t$

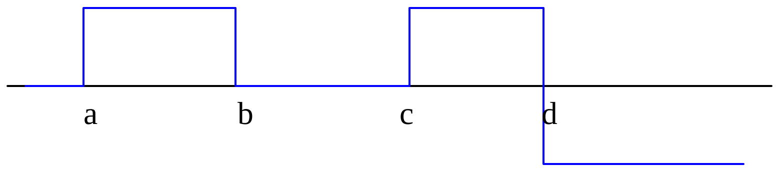
  - $u(t) = \int_{-\infty}^{t} d(t)dt$

    $= \int d(t-s)d(t-s) \;; s = t-t$

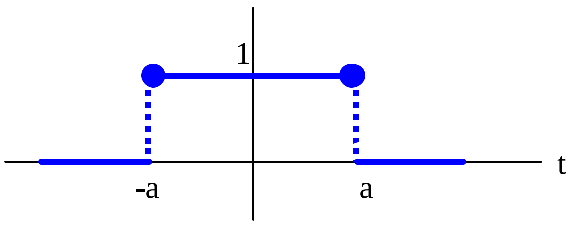
    $= \int_{\infty}^{0} d(t-s)(-d(s)) = \int_{0}^{\infty} d(t-s)d(s)$

  - $ku(t) = \int_{-\infty}^{t} kd(t)dt$

- $u(t-a)-u(t-b)+u(t-c)-2u(t-d)$



- **rectangular pulse** $P_a(t)=\begin{cases}1 & |t|\le a\\ 0 & |t|>a\end{cases}$



- $d(t)=$ **unit impulse** = Dirac ***d***-function

$$=\begin{cases}0 & t\neq 0\\ \infty & t=0\end{cases}$$

$$=\frac{d}{dt}u(t)=\lim_{a\to 0}\frac{u\left(t+\frac{a}{2}\right)-u\left(t-\frac{a}{2}\right)}{a}$$

$$=\lim_{a\to 0}\frac{1}{a}p_{\frac{a}{2}}(t)$$

- $d(t-t_0)$ = unit impulse occurring at time t_0
- $\int d(t)[\cdots]dt$ means $\lim_{a\to 0}\int_{-\infty}^{\infty}\frac{1}{a}p_{\frac{a}{2}}(t)[\cdots]dt$
- $td(t) = 0$ "t

- $\int_{-\infty}^{\infty} d(t)\,dt=1$

$$\int_{-\infty}^{\infty} d(t)dt=\lim_{a\to 0}\frac{1}{a}\int_{-\infty}^{\infty}p_{\frac{a}{2}}(t)dt=\lim_{a\to 0}\frac{1}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}}dt=1$$

- $\int_{-\infty}^{\infty} d(t-t_0)f(t)dt=f(t_0)$

$$\int_{-\infty}^{\infty} d(t-t_0) f(t) dt = \int_{-\infty}^{\infty} d(t) f(t+t_0) dt = \lim_{a\to 0} \frac{1}{a} \int_{-\infty}^{\infty} p_{\frac{a}{2}}(t) f(t+t_0) dt$$

$$= \lim_{a\to 0} \frac{1}{a} \int_{-\frac{a}{2}}^{\frac{a}{2}} f(t+t_0) dt = \lim_{a\to 0} \frac{1}{a} \int_{t_0-\frac{a}{2}}^{t_0+\frac{a}{2}} f(t) dt = f(t_0)$$

•	$\int_a^b d(x-x_0) g(x) dx = \begin{cases} g(x_0) & a < x_0 < b \\ 0 & otherwise \end{cases}$
•	$d(t) * x(t) = x(t)$

$$\int_{-\infty}^{\infty} d(t-t) x(t) dt = \lim_{a\to 0} \int_{-\infty}^{\infty} \frac{1}{a} p_{\frac{a}{2}}(t-t) x(t) dt = \frac{1}{a} \lim_{a\to 0} \int_{t-\frac{a}{2}}^{t+\frac{a}{2}} x(t) dt = x(t)$$

•	$d(t-T) * x(t) = x(t-T) \Rightarrow$ pure delay system

$$\int_{-\infty}^{\infty} d(t-T-t) x(t) dt = \lim_{a\to 0} \int_{-\infty}^{\infty} \frac{1}{a} p_{\frac{a}{2}}(t-T-t) x(t) dt = \frac{1}{a} \lim_{a\to 0} \int_{(t-T)-\frac{a}{2}}^{(t-T)+\frac{a}{2}} x(t) dt = x(t-T)$$

- $d(t) \xrightarrow{S} h(t)$

Signal properties

•	**Right-sided** $\leftrightarrow \exists t_0 \mid x(t) = 0$ when $t < t_0$, "t **Left-sided** $\leftrightarrow \exists t_1 \mid x(t) = 0$ when $t > t_1$, "t
•	**Time-limited** $\leftrightarrow \exists (t_0, t_1) \mid x(t) = 0$ for $t < t_0$ and $t > t_1$ $\leftrightarrow$ right-sided and left-sided
•	**Causal** $\leftrightarrow x(t) = 0 \; \forall t < 0$ **Anti-causal** $\leftrightarrow x(t) = 0 \; \forall t > 0$

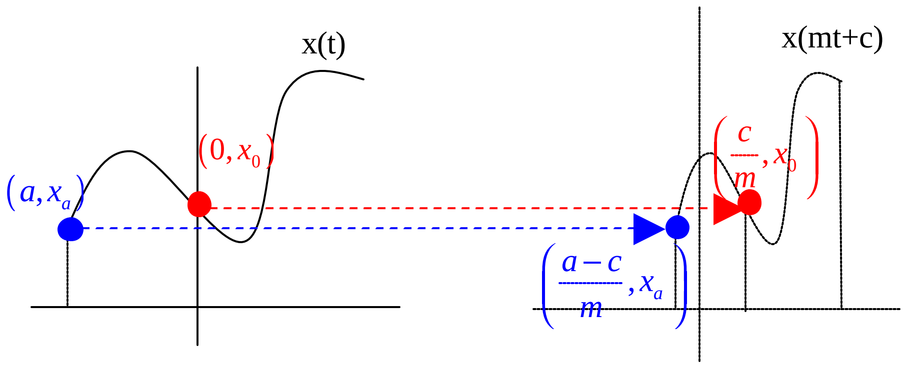
Time -shifting / -scaling (-dilated or -compressed) / -reversal

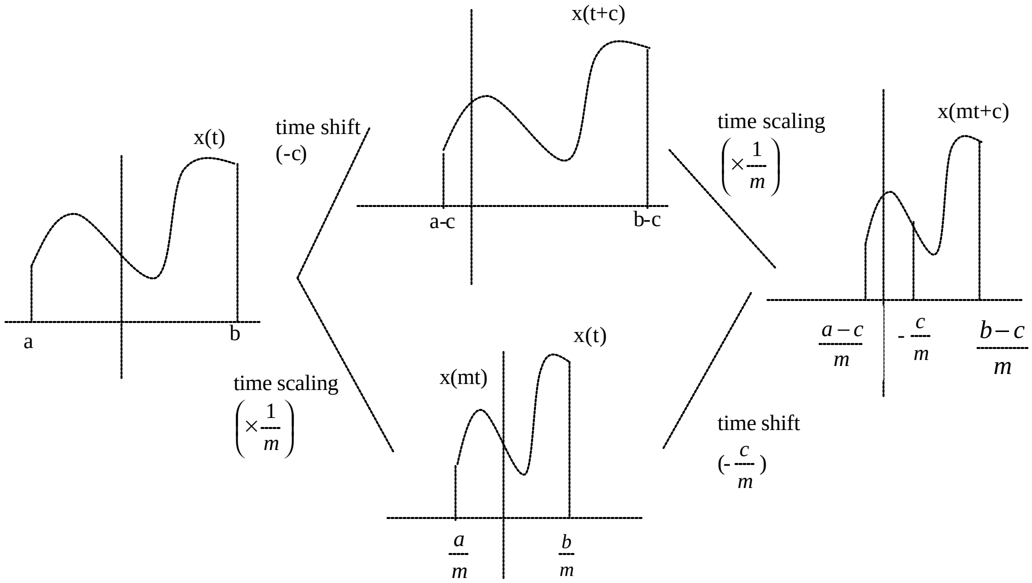
- $x(t)$ ® $\tilde{x}(t) = x(mt+c)$
  - Point $(a, x_a) \rightarrow \left(\frac{a-c}{m}, x_a \right)$ because to have $mt + c = a$, need $t = \frac{a-c}{m}$

    Find t' such that $\tilde{x}(t') = x(mt'+c) = x(a) = x_a$

    $\therefore mt'+c = a \rightarrow t' = \frac{a-c}{m}$

Point $(0, x_0) \rightarrow \left(\frac{c}{m}, x_0\right)$





Convolution

- **convolution** of $x_1(t)$ and $x_2(t)$

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(t)x_2(t-t)dt = \int_{-\infty}^{\infty} x_1(t-t)x_2(t)dt \ ; -\infty < t < \infty$$

- Commutative: $\int_{-\infty}^{\infty} x_1(t)x_2(t-t)dt = \int_{-\infty}^{\infty} x_1(t-t)x_2(t)dt$

  let $V = t - t$

$$\int_{-\infty}^{\infty} x_1(t)x_2(t-t)dt = \int_{\infty}^{-\infty} x_1(t-z)x_2(z)(-dz) = \int_{-\infty}^{\infty} x_1(t-z)x_2(z)dz$$

- May not be well-defined

- If $x_1(t)$ and $x_2(t)$ are both causal

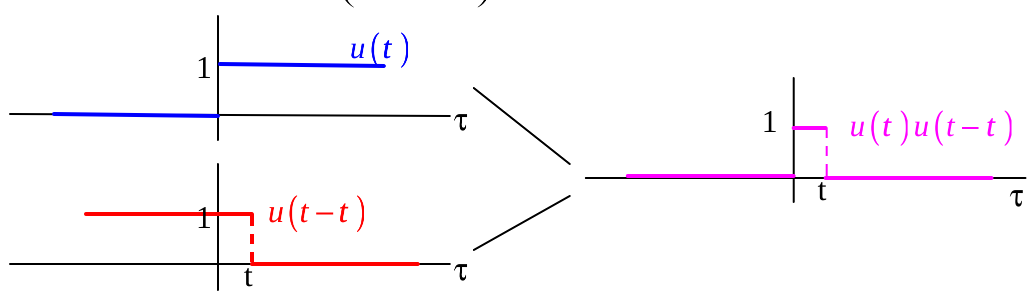
  then $x_1(t) * x_2(t)$ always exist and is also causal

- Useful identity
  - $\int_{-\infty}^{\infty} f(t)u(t)u(a-t)dt = \left(\int_0^a f(t)dt\right)u(a)$
- Example
  - $u(t)* f(t)u(t) = \left(\int_0^t f(t)dt\right)u(t)$

$$\int_{-\infty}^{\infty} f(t)u(t)u(t-t)dt = \int_0^{\infty} f(t)u(t-t)dt \quad \text{because of} \quad u(t)$$

$$= \begin{cases} \int_0^t f(t)dt & t \geq 0 \\ 0 & t < 0 \end{cases} \quad \text{because of} \quad u(t\text{-}t)$$

$$= \left(\int_0^t f(t)dt\right)u(t)$$



- $x_1(t) = u(t\text{-}t_0),\ x_2(t) = f(t)u(t)$

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} u(t-t-t_0)\, f(t)u(t)\, dt = \int_{-\infty}^{\infty} u((t-t_0)-t)\, f(t)u(t)\, dt$$

$$= \left(\int_0^{t-t_0} f(t)dt\right)u(t-t_0) \quad ; \int_{-\infty}^{\infty} f(t)u(t)u(a-t)dt = \left(\int_0^a f(t)dt\right)u(a)$$

$$= \left(\int_{t_0}^{t} f(m-t_0)dm\right)u(t-t_0) \quad ; m = t + t_0$$

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} u(t-t_0)\, f(t-t)u(t-t)\, dt$$

$$= \int_{t_0}^{\infty} u(t-t)\, f(t-t)dt$$

$$= \left(\int_{t_0}^{t} f(t-t)dt\right)u(t-t_0)$$

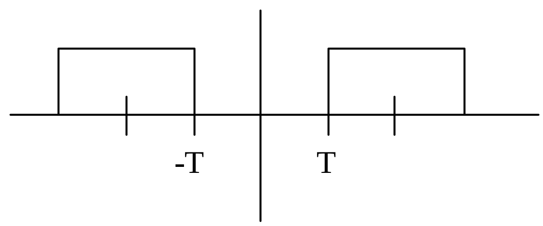
- *$x_1(t)$ = $u(t)$, $x_2(t)$ = $f(t)u(-t)$*

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} u(t-t)\, f(t) u(-t)\, dt \;= \int_{-\infty}^{0} u(t-t)\, f(t) dt$$

$$= \begin{cases} \int_{-\infty}^{0} f(t) dt & ;t \geq 0 \\ \int_{-\infty}^{t} f(t) dt & ;t < 0 \end{cases}$$

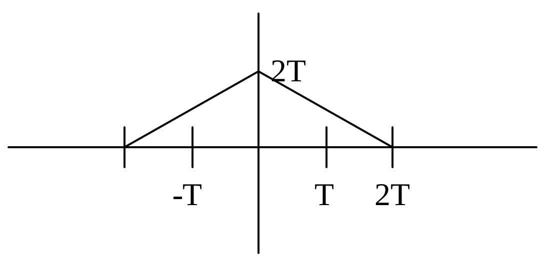
- $$p_T(t) * p_T(t) = \int_{-\infty}^{\infty} p_T(t)\, p_T(t-t) dt = \int_{-T}^{T} p_T(t-t) dt$$

= 0 for $|t| \geq 2T$



= 2T-|t| for $0 \leq |t| < 2T$

$$p_T(t) * p_T(t) = \begin{cases} 0 & |t| \geq 2T \\ 2T - |t| & 0 \leq |t| < 2T \end{cases}$$



- RC Series. V_s = *w(t)*, $V_c = y(t)$

$$V_s = V_R + V_C = R\left(C \frac{d}{dt} V_C \right) + V_C$$

$$w(t) = RCy'(t) + y(t)$$

$$aw(t) = y'(t) + ay(t) \quad ; a = \frac{1}{RC}$$

Let $z(t) = e^{at} y(t)$

$$z'(t) = e^{at} y'(t) + ae^{at} y(t) = e^{at}\left(y'(t) + ay(t)\right) = aw(t) e^{at}$$

$$z(t) = \int_0^t aw(t) e^{at} dt \quad t \geq 0, z(0) = y(0) = 0$$

$$y(t) = e^{-at} z(t) = \int_0^t aw(t)e^{-a(t-t)}dt \quad ,t \ge 0$$

$$= \int_{-\infty}^{t} aw(t)e^{-a(t-t)}dt \quad ,t \ge 0 \; ; w(t<0)=0$$

$$= \left(\int_{-\infty}^{t} \left[ae^{-a(t-t)} \right] w(t)\,dt \right) u(t) \; ,all \;\; t \; ;y \; (t<0)=0$$

$$= \int_{-\infty}^{\infty} \left[ae^{-a(t-t)} u(t-t) \right] w(t)\,dt$$

$$= \left(ae^{-at} u(t) \right) * \left(w(t) \right)$$

- $x_1(t)*x_2(t) = y(t) \Rightarrow \; x(t-t_1)*x(t-t_2) = y(t-t_1-t_2)$

$$x_1(t-t_1)*x_2(t-t_2) = w_1(t)*w_2(t) = \int_{-\infty}^{\infty} w_1(t)w_2(t-t)dt$$

$$= \int_{-\infty}^{\infty} x_1(t-t_1)x_2\left((t-t)-t_2\right)dt$$

$$= \int_{-\infty}^{\infty} x_1(m)x_2\left(\left(t-(m+t_1)\right)-t_2\right)dm \quad ;m = t - t_1, dm = dt$$

$$= \int_{-\infty}^{\infty} x_1(m)x_2\left(\left(t-(t_1+t_2)\right)-m\right)dm$$

- $d(t)*x(t) = x(t)$

  $d(t-T)*x(t) = x(t-T)$

SISO

- SISO $\Rightarrow$ single-input single-output system

- **Memoryless**

  A system is memoryless if its output for each value of the independent variable at a given time is dependent only on the input at the same time

- **Invertible**

  A system is invertible if distinct inputs lead to distinct outputs

  - *x(t)* —System→ *y(t)* —Inverse system→ *w(t)* = *x(t)*

- **Causality**

  A system is causal if the output at any time depends only on values of the input at the present time and in the past. $\Rightarrow$ Nonanticipative

  - Memoryless → causal

- Any SISO system that models a physically buildable object is causal

- **Stability**

  a stable system is one in which small inputs lead to response that do not diverge

- If the input to a stable system is bounded (i.e. if its magnitude does not grow without bound), then the output must also be bounded and therefore cannot diverge

- **Time invariance**

  A system is time invariant if the behavior and characteristics of the system are fixed over time

- A system is time invariant if a time shift in the input signal (replace t by $t+t_0$) results in an identical time shift in the output signal

  $w(t) \xrightarrow{S} y(t) \Rightarrow \tilde{w}(t) = w(t+t_0) \xrightarrow{S} \tilde{y}(t) = y(t+t_0)$

- To test

  ① $y(t)$

  ② Find $\tilde{y}$ (t) from system's def. and $\tilde{w}(t) = w(t+t_0)$

  ③ Compare $y(t+t_0) = \tilde{y}$ (t)?

- Example of time-invariant system

  - $y(t) = F(w(t))$

    $$w(t) \xrightarrow{S} F(w(t)) = y(t)$$

    $$\tilde{w}(t) = w(t+t_0) \xrightarrow{S} F(\tilde{w}(t)) = F(w(t+t_0)) = \tilde{y}(t)$$

    $$y(t+t_0) = F(w(t+t_0)) = \tilde{y}(t)$$

  - $y(t) = w(f(t))$ when $w(f(t)+t_0) = w(f(t+t_0))$

    $$w(t) \xrightarrow{S} w(f(t)) = y(t)$$

    $$\tilde{w}(t) = w(t+t_0) \xrightarrow{S} \tilde{w}(f(t)) = w(f(t)+t_0) = \tilde{y}(t)$$

    $$y(t+t_0) = w(f(t+t_0))$$

    For example, when $f(t) = t + a$

  - $y(t) = \sum_k F_k(w(t-t_k))$

    $$w(t) \xrightarrow{S} \sum_k F_k(w(t-t_k)) = y(t)$$

    $$\tilde{w}(t) = w(t+t_0) \xrightarrow{S} \sum_k F_k(\tilde{w}(t-t_k)) = \sum_k F_k(w(t-t_k+t_0)) = \tilde{y}(t)$$

    $$y(t+t_0) = \sum_k F_k(w(t+t_0-t_k)) = \tilde{y}(t)$$

- $y(t) = \int_{t_1}^{t_2} F(w(t-\tau))d\tau$

$$w(t) \xrightarrow{S} \int_{t_1}^{t_2} F(w(t-\tau))d\tau = y(t)$$

$$\tilde{w}(t) = w(t+t_0) \xrightarrow{S} \int_{t_1}^{t_2} F(\tilde{w}(t-\tau))d\tau = \int_{t_1}^{t_2} F(\tilde{w}(t+t_0-\tau))d\tau = \tilde{y}(t)$$

$$y(t+t_0) = \int_{t_1}^{t_2} F(w(t+t_0-\tau))d\tau = \tilde{y}(t)$$

- Example of time-varying system
  - $y(t) = F(t, w(t))$

$$w(t) \xrightarrow{S} F(t, w(t)) = y(t)$$

$$\tilde{w}(t) = w(t+t_0) \xrightarrow{S} F(t, \tilde{w}(t)) = F(t, w(t+t_0))$$

$$y(t+t_0) = F(t+t_0, w(t+t_0)) \neq \tilde{y}(t)$$

  - $y(t) = F(t)G(w(t))$

$$w(t) \xrightarrow{S} F(t)G(w(t)) = y(t)$$

$$\tilde{w}(t) = w(t+t_0) \xrightarrow{S} F(t)G(\tilde{w}(t)) = F(t)G(w(t+t_0)) = \tilde{y}(t)$$

$$y(t+t_0) = F(t+t_0)G(w(t+t_0)) \neq \tilde{y}(t)$$

  - $y(t) = w(f(t))$ when $w(f(t)+t_0) \neq w(f(t+t_0))$

    For example, when $f(t) = at$

- **linearity**

$w(t) = c_1 w_1(t) + c_2 w_2(t) \xrightarrow{S} y(t) = c_1 y_1(t) + c_2 y_2(t)$

or

$\sum_k c_k x_k(t) \xrightarrow{S} \sum_k c_k y_k(t)$: superposition property

- $0 \xrightarrow{\text{linear system}} 0$
- Example of linear system
  - $y(t) = f(t)w(g(t))$

$$w_1(t) \xrightarrow{S} f(t)w_1(g(t)) = y_1(t)$$

$$w_2(t) \xrightarrow{S} f(t)w_2(g(t)) = y_2(t)$$

$$c_1 w_1(t) + c_2 w_2(t) \xrightarrow{S} f(t)(c_1 w_1(g(t)) + c_2 w_2(g(t))) = c_1 y_1(t) + c_2 y_2(t)$$

- $y(t) = f(t)w(t)$
- $y(t) = a \cdot w(t)$

- Example of nonlinear system
  - $y(t) = a \cdot w(t) + b$ when $b \neq 0$

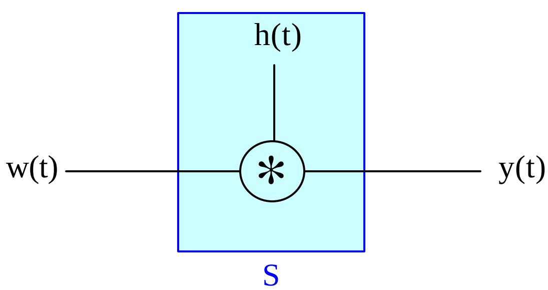
$$w_1(t) \xrightarrow{S} a \cdot w_1(t) + b = y_1(t)$$
$$w_2(t) \xrightarrow{S} a \cdot w_2(t) + b = y_2(t)$$
$$c_1 y_1(t) + c_2 y_2(t) = ac_1 \cdot w_1(t) + bc_1 + ac_2 \cdot w_2(t) + bc_2$$
$$= a\left(c_1 \cdot w_1(t) + c_2 \cdot w_2(t)\right) + b(c_1 + c_2)$$
$$c_1 w_1(t) + c_2 w_2(t) \xrightarrow{S} a \cdot \left(c_1 w_1(t) + c_2 w_2(t)\right) + b \neq c_1 y_1(t) + c_2 y_2(t)$$

Unless $b(c_1 + c_2) = b \Rightarrow b(c_1 + c_2 - 1) = 0 \Rightarrow b = 0$

LTI SISO system

- LTI = Linear, time-invariant

- $w(t) \xrightarrow{S} y(t) = h(t) * w(t) = \int_{-\infty}^{\infty} h(t - t)w(t)dt$



- Convolutional system $\left(w(t) \xrightarrow{S} y(t) = h(t) * w(t)\right)$ is LTI
- Every reasonable LTI system is (or can be thought of as being) of this "convolutional" type!

- To see this,
  - Linearity of convolutional system

$$w_1(t) \xrightarrow{S} y_1(t) = h(t) * w_1(t) = \int_{-\infty}^{\infty} h(t - t)w_1(t)dt$$

$$w_2(t) \xrightarrow{S} y_2(t) = h(t) * w_2(t) = \int_{-\infty}^{\infty} h(t - t)w_2(t)dt$$

$$w(t) = c_1 w_1(t) + c_2 w_2(t) \xrightarrow{S} y(t) = \int_{-\infty}^{\infty} h(t - t)w(t)dt$$

$$y(t)=\int_{-\infty}^{\infty} h(t-t)\left(c_1 w_1(t)+c_2 w_2(t)\right)dt$$

$$=c_1\int_{-\infty}^{\infty} h(t-t)w_1(t)dt+c_2\int_{-\infty}^{\infty} h(t-t)w_2(t)dt$$

$$=c_1 y_1(t)+c_2 y_2(t)$$

- Time-invariance of convolutional system

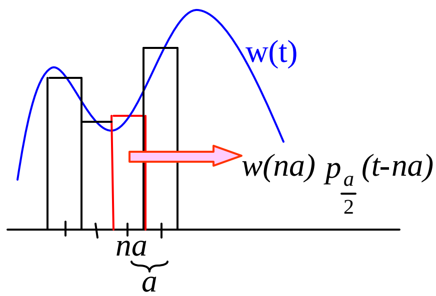
$$\tilde{w}(t)=w(t+t_0)\xrightarrow{S}\tilde{y}(t)=\int_{-\infty}^{\infty} h(t-t)\tilde{w}(t)dt=\int_{-\infty}^{\infty} h(t-t)w(t+t_0)dt$$

$$y(t+t_0)=\int_{-\infty}^{\infty} h((t+t_0)-t)w(t)dt=\int_{-\infty}^{\infty} h(t-(t-t_0))\,w(t)\,dt$$

$$=\int_{-\infty}^{\infty} h(t-m)w(m+t_0)dm \quad ;m=t-t_0,t=m+t_0,dm=dt$$

$$=\tilde{y}(t)$$

- Every reasonable LTI system is (or can be thought of as being) of "convolutional" system



$$w(t)=\lim_{a\to 0}\left(\sum_{n=-\infty}^{\infty} w(na)\,p_{\frac{a}{2}}(t-na)\right)$$

$$y(t)=\lim_{a\to 0}\left(\begin{matrix}response\\ to\end{matrix}\left(\sum_{n=-\infty}^{\infty} w(na)p_{\frac{a}{2}}(t-na)\right)\right)$$

$$=\lim_{a\to 0}\left(\sum_{n=-\infty}^{\infty} w(na)\cdot\left(\begin{matrix}response\\ to\end{matrix}\left(\frac{1}{a}\,p_{\frac{a}{2}}(t-na)\right)\right)\cdot a\right);\text{linear}$$

$$=\lim_{a\to 0}\left(\sum_{n=-\infty}^{\infty} w(na)\cdot h_a(t-na)\cdot a\right);\text{time}-\text{invariance}$$

$$=\int_{-\infty}^{\infty} h(t-t)w(t)\,dt \quad ;\lim_{a\to 0} a=dt$$

$$=h(t)*w(t)$$

- **Impulse response of the system** = $h(t)$
  = the response of the system to an impulse
  - $\delta(t) \xrightarrow{S} h(t)$

  Use this to find $h(t)$ when the system is explicitly defined.
  - Example
    - $y(t) = F\left(w\left(f(t)\right)\right) \Rightarrow h(t) = F\left(d\left(f(t)\right)\right)$
  - $$h(t) = \lim_{a \to 0} \underset{to}{response}\left\{\frac{1}{a} p_{\frac{a}{2}}(t)\right\} = \underset{to}{response} \lim_{a \to 0}\left\{\frac{1}{a} p_{\frac{a}{2}}(t)\right\} = \underset{to}{response} d(t)$$

- $h(t) = \dfrac{d}{dt} y_s(t)$

  By time-invariance,

  $$u\left(t + \frac{a}{2}\right) \xrightarrow{S_{LTI}} y_s\left(t + \frac{a}{2}\right)$$
  $$u\left(t - \frac{a}{2}\right) \xrightarrow{S_{LTI}} y_s\left(t - \frac{a}{2}\right)$$

  By linearity

  $$\frac{u\left(t + \frac{a}{2}\right) - u\left(t - \frac{a}{2}\right)}{a} \xrightarrow{S_{LTI}} \frac{y_s\left(t + \frac{a}{2}\right) - y_s\left(t - \frac{a}{2}\right)}{a}$$

  $$\lim_{a \to 0} \frac{u\left(t + \frac{a}{2}\right) - u\left(t - \frac{a}{2}\right)}{a} \xrightarrow{S_{LTI}} \lim_{a \to 0} \frac{y_s\left(t + \frac{a}{2}\right) - y_s\left(t - \frac{a}{2}\right)}{a}$$

  Therefore, $d(t) \xrightarrow{S_{LTI}} h(t) = \dfrac{d}{dt} y_s(t)$

- **LTI system's step response** $\Rightarrow y_s(t)$
  $u(t) \xrightarrow{S} y_s(t) = u(t)*h(t)$

- Causal system $\leftrightarrow$ $h(t)$ is causal

  $h(t)$ is causal $\Rightarrow$ Causal system

  $h(t)$ is causal $\Rightarrow h(t) = 0$ when $t < 0 \Rightarrow h(t\text{-}t) = 0$ when $t > t$

  $$y(t) = \int_{-\infty}^{\infty} h(t-t)w(t)dt = \int_{-\infty}^{t} h(t-t)w(t)dt$$

  Therefore, $y(t)$ depend only on $w(t)$ for t £ t, not $t > t$ when $h(t)$ is causal

- $\cos(t) \xrightarrow{S} \text{Re}\left\{h(t) * e^{jt}\right\}$ if $h(t)$ is real

$$\cos(t) = \frac{1}{2}\left(e^{jt} + e^{-jt}\right)$$

$$\cos(t) * h(t) = \frac{1}{2}\left(h(t) * e^{jt} + h(t) * e^{-jt}\right)$$

$$= \frac{1}{2}\left(h(t) * e^{jt} + \overline{h(t) * e^{jt}}\right) = \mathrm{Re}\left\{h(t) * e^{jt}\right\}$$

- $\cos(t)u(t) * h(t) \xrightarrow{S} \mathrm{Re}\left\{h(t) * e^{jt}u(t)\right\}$ if *h(t)* is real

$$\cos(t)u(t) * h(t) = \frac{1}{2}\left(h(t) * e^{jt}u(t) + h(t) * e^{-jt}u(t)\right)$$

$$= \frac{1}{2}\left(h(t) * e^{jt}u(t) + \overline{h(t) * e^{jt}u(t)}\right) = \mathrm{Re}\left\{h(t) * e^{jt}u(t)\right\}$$

- $\sin(t) \xrightarrow{S} \mathrm{Re}\left\{h(t) * \left(e^{jt}u(t)\right)\right\}$

$$\sin(t) * h(t) = \frac{1}{2j}\left(h(t) * e^{jt} - h(t) * e^{-jt}\right)$$

$$= \frac{1}{2j}\left(h(t) * e^{jt} - \overline{h(t) * e^{jt}}\right) = \mathrm{Im}\left\{h(t) * e^{jt}\right\}$$

- $w(t) \xrightarrow{S_1} \xrightarrow{S_2} y(t) = w(t) * \underbrace{\left(h_1(t) * h_2(t)\right)}_{h(t)}$

LTI and $\mathcal{F}$-transform

- **Frequency response** of the system: $\hat{H}(w) = \int_{-\infty}^{\infty} h(t) e^{-jwt} dt$
- $e^{jwt} \xrightarrow{S_{LTI}} \hat{H}(w) e^{jwt}$

$$e^{jwt} * h(t) = \int_{-\infty}^{\infty} h(t) e^{jw(t-t)} dt = e^{jwt}\left(\int_{-\infty}^{\infty} h(t) e^{-jwt} dt\right)$$

  - $e^{jwt} \xrightarrow{S} (\text{constant}) e^{jwt}$
  - $e^{jkw_0 t} \xrightarrow{S} \hat{H}(kw_0) e^{jkw_0 t}$

- $y(t) = h(t) * w(t) \underset{\Im^{-1}}{\overset{\Im}{\rightleftharpoons}} \hat{Y}(w) = \hat{H}(w) \cdot \hat{W}(w)$

- To find $\hat{H}(w)$
  - Use $\int_{-\infty}^{\infty} h(t) e^{-jwt} dt$, if know h(t)
  - Use $e^{jwt} \xrightarrow{S} \hat{H}(w) e^{jwt}$ if given implicit equation

- Example
  - $$\frac{d}{dt}y(t)+py(t)=qw(t)$$

    Let $w(t) = e^{jwt} \Rightarrow y(t) = \hat{H}(w)e^{jwt}$

    $$\frac{d}{dt}y(t)+py(t)=qw(t)$$

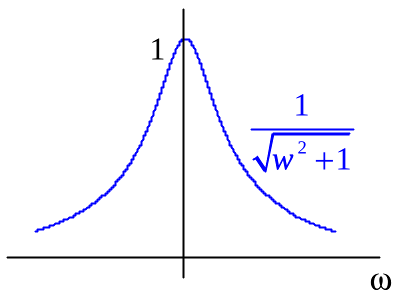
    $$\frac{d}{dt}\left(\hat{H}(w)e^{jwt}\right)+p\hat{H}(w)e^{jwt}=qe^{jwt}$$

    $$\hat{H}(w)(jw)\cancel{e^{jwt}}+p\hat{H}(w)\cancel{e^{jwt}}=q\cancel{e^{jwt}}$$

    $$\hat{H}(w)(jw+p)=q$$

    $$\hat{H}(w)=\frac{q}{jw+p}$$

    $$\left|\hat{H}(w)\right|=\frac{|q|}{\sqrt{w^2+p^2}} \Rightarrow \text{low-pass filter}$$

  - $$\frac{d}{dt}y(t)+py(t)=\frac{d}{dt}w(t)$$

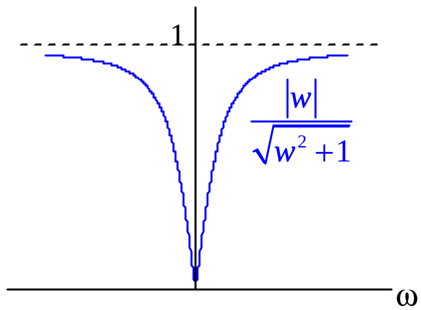
    Let $w(t) = e^{jwt} \Rightarrow y(t) = \hat{H}(w)e^{jwt}$

    $$\frac{d}{dt}y(t)+py(t)=\frac{d}{dt}w(t)$$

    $$jw\hat{H}(w)\cancel{e^{jwt}}+p\hat{H}(w)\cancel{e^{jwt}}=jw\cancel{e^{jwt}}$$

    $$\hat{H}(w)=\frac{jw}{jw+p}$$

    $$\left|\hat{H}(w)\right|=\frac{|w|}{\sqrt{w^2+p^2}} \Rightarrow \text{high-pass filter}$$



- Given w(t) and y(t), $\hat{H}(w) = \dfrac{\hat{Y}(w)}{\hat{W}(w)}$

- When finding $\left|\hat{H}(w)\right|$ (magnitude), don't forget take the absolute value of the result
- Example: pure delay system

  $w(t) \xrightarrow{S_{LTI}} y(t) = w(t\text{-}T)$; $T > 0 \Rightarrow h(t) = d(t\text{-}T)$

  Proof 1

  $$\because\ d(t-T) * w(t) = w(t-T)$$

  Proof 2

  Let $w(t) = d(t)$, then the output will be $d(t\text{-}T)$.

  However, also know that the output is $w(t)*h(t) = d(t)*h(t) = h(t)$.

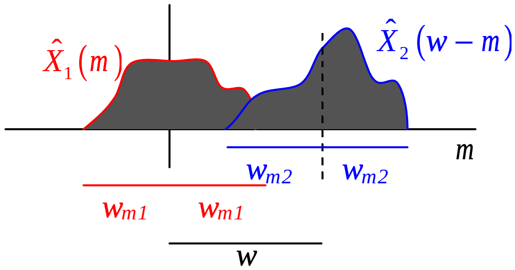
  Thus, $h(t) = \delta(t\text{-}T)$.
- w_m = bandwidth of $x(t)$; $x(t)$ is w_m-band-limited $\Rightarrow$ $\hat{X}(|w| > w_m) = 0$
  - $x_1(t) + x_2(t) \Rightarrow = \max(w_{m1}, w_{m2})$

    To see this, $x_1(t) + x_2(t) \underset{\Im^{-1}}{\overset{\Im}{\rightleftharpoons}} \hat{X}_1(w) + \hat{X}_2(w)$
  - $x_1(t)*x_2(t) \Rightarrow = \min(w_{m1}, w_{m2})$

    To see this, $x_1(t) * x_2(t) \underset{\Im^{-1}}{\overset{\Im}{\rightleftharpoons}} \hat{X}_1(w)\hat{X}_2(w)$
  - $x_1(t)\, x_2(t) \Rightarrow = w_{m1} + w_{m2}$

    To see this, $x_1(t)\, x_2(t) \underset{\Im^{-1}}{\overset{\Im}{\rightleftharpoons}} \dfrac{1}{2p} \hat{X}_1(w) * \hat{X}_2(w)$

    $$\hat{X}_1(w) * \hat{X}_2(w) = \int_{-\infty}^{\infty} \hat{X}_1(m)\hat{X}_2(w-m)\,dm$$



Need $w > w_{m1} + w_{m2}$ to ensure no overlapping region, and thus the multiplication gives 0 at every *m*, yielding zero integral.

LTI and $\mathcal{L}$-transform

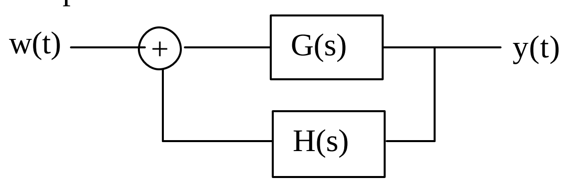
- $h(t) \overset{\mathcal{L}}{\rightleftharpoons} H(s)\ \ (ROC)_H$

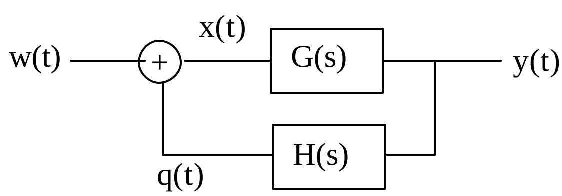
  impulse response $\overset{\mathcal{L}}{\rightleftharpoons}$ transfer function
- $y(t) = w(t) * h(t) \overset{\mathcal{L}}{\rightleftharpoons} Y(S) = W(S) \cdot H(S)$

  $(ROC)_Y = (ROC)_H \cdot (ROC)_W$
- $w(t) = e^{s_0 t} \xrightarrow{S_{LTI}} y(t) = H(s_0) e^{s_0 t}$ if $s_0 \in (ROC)_H$
- For *x(t)* to be **causal** and have *X(s)* as "the formula part" of its $\mathcal{L}$-transform, need all poles of *X(s)* to be to the left of $(ROC)_X \Rightarrow$ have only u(t)-terms, no u(-t)-term
- For 1) rational H(s) 2) causal system,
  $(ROC)_H$ = the part of complex plane to the right of all poles of H(s)

Stability of causal LTI system

- A causal system is **BIBO stable** $\leftrightarrow$
  - $\forall$ bounded *w(t)* $\xrightarrow{S_{LTI}}$ well defined *y(t)* that is also bounded
  - *h(t)* is absolutely integrable $\int_{-\infty}^{\infty} |h(t)| dt = \left(\int_{0}^{\infty} |h(t)| dt < \infty \right)$
  - all poles of rational *H(s)* lies in Re{s} < 0
    - so that $\lim_{t \to \infty} t^{k_i} e^{s_{0i} t} u(t) = 0 \rightarrow \int_{0}^{\infty} |h(t)| dt$ is finite.
- Example





$$\left.\begin{aligned} X(s) &= W(s) + Q(s) \\ Q(s) &= H(s) Y(s) \end{aligned}\right\} \Rightarrow X(s) = W(s) + H(s) Y(s)$$

$$Y(s) = X(s)G(s) = \left(W(s) + H(s)Y(s)\right)G(s)$$
$$= W(s)G(s) + H(s)Y(s)G(s) = \underbrace{\frac{G(s)}{1 - H(s)G(s)}}_{T(s)} W(s) \text{ ??}$$

Find Re{poles} of *T(s)*, unstable when this > 0

- **General Encirclement Rule**

  Assume

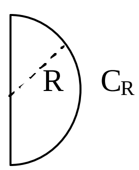
  - C is a clockwise-directed closed curve in the complex plane
  - *F(s)* is a rational function that has no pole/zero on C.

  Then

  The net number of times that closed *F(C)* encircles "0" clockwise

  = {#zeros of *F(s)* enclosed by *C*} – {#poles of *F(s)* enclosed by *C*}
  counting multiplicity

- The **Nyquist locus**
  - Assumption:
    - G(s) strictly proper and H(s) proper rational
    - All poles of G(s) and H(s) lie in Re{s} < 0
    - G(jw)H(jw) ≠ -1 for any real w
  - ⇒ directed curve using F(s) = G(s)H(s) to map the upward-directed imaginary axis
  - ⇒ set of all points of the form F(jω) = G(jω)H(jω) for increasing $-\infty < \omega < \infty$
  - ⇒ $\lim_{R \to \infty} F(C_R)$



  - $\lim_{|s| \to \infty} F(s) = 0$
  - Nyquist locus begins ($\omega \to -\infty$) and ends ($\omega \to \infty$) @ 0

- **Nyquist Criterion** (restricted version):
  - # poles of *T(s)* that lie in Re{s} ≥ 0

    = the net #times that F(jω) with increasing ω encircles the point s = -1, clockwise
  - The feed back system is stable ↔ *T(s)* has no pole in Re{s} ≥ 0

  ≡ Nyquist locus makes <u>no net clockwise encirclements of "s = -1"</u>