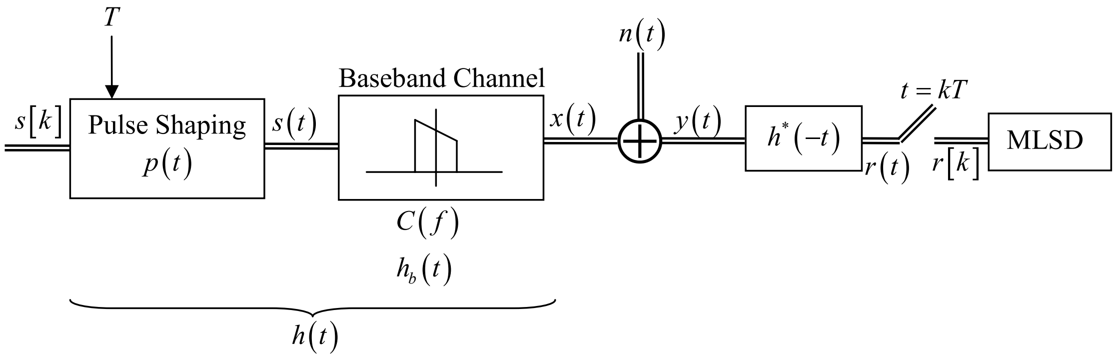
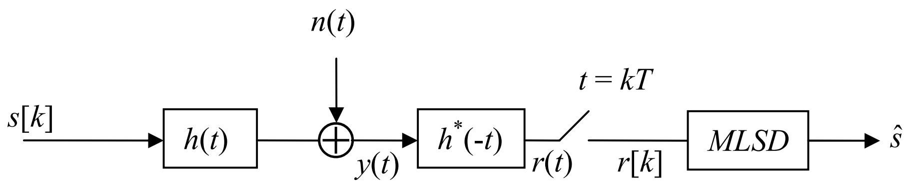
- The Baseband Model



- $h(t) = p(t) * h_b(t)$

MLSD

- The Optimal Receiver



MLSD

- Let $y(t) = \sum_{k=0}^{N-1} s[k]h(t-kT) + n(t)$, where $n(t)$ is complex AWGN with PSD N_0, then

  $\hat{S}_{ML} = \arg\min_{\bar{s}} -2\,\mathrm{Re}\left\{\sum_{k=0}^{N-1} s^*[k]r[k]\right\} + \left(\sum_{k=0}^{N-1}\sum_{i=0}^{N-1} s[k]s^*[i]\rho[i-k]\right)$, where

  $r[k] = y(t) * h^*(-t)\big|_{t=kT}$

  $\rho[k] = h(t) * h^*(-t)\big|_{t=kT}$

Proof:

$$\hat{S}_{ML} = \arg\min_{\vec{s}} \int \left| y(t) - \sum_{k=0}^{N-1} s[k] h(t-kT) \right|^2 dt$$
$$= \arg\min_{\vec{s}} \int \left(y(t) - \sum_{k=0}^{N-1} s[k] h(t-kT) \right) \left(y^*(t) - \sum_{k=0}^{N-1} s^*[k] h^*(t-kT) \right) dt$$
$$= \arg\min_{\vec{s}} -2\,\mathrm{Re}\left\{ \int y(t) \sum_{k=0}^{N-1} s^*[k] h^*(t-kT) dt \right\} + \int \left(\sum_{k=0}^{N-1} s[k] h(t-kT) \right) \left(\sum_{i=0}^{N-1} s^*[i] h^*(t-iT) \right) dt$$
$$= \arg\min_{\vec{s}} -2\,\mathrm{Re}\left\{ \int y(t) \sum_{k=0}^{N-1} s^*[k] h^*(t-kT) dt \right\} + \int \left(\sum_{k=0}^{N-1} \sum_{i=0}^{N-1} s^*[i] h^*(t-iT) s[k] h(t-kT) \right) dt$$
$$= \arg\min_{\vec{s}} -2\,\mathrm{Re}\left\{ \int y(t) \sum_{k=0}^{N-1} s^*[k] h^*(t-kT) dt \right\} + \int \left(\sum_{k=0}^{N-1} \sum_{i=0}^{N-1} s[k] s^*[i] h(t-kT) h^*(t-iT) \right) dt$$
$$= \arg\min_{\vec{s}} -2\,\mathrm{Re}\left\{ \sum_{k=0}^{N-1} s^*[k] \left(\int h^*(t-kT) y(t) dt \right) \right\} + \int \left(\sum_{k=0}^{N-1} \sum_{i=0}^{N-1} s[k] s^*[i] h(t-kT) h^*(t-iT) \right) dt$$

Let $r(t) = y(t) * h^*(-t) = \int y(t) h^*(-(t-\tau)) d\tau = \int y(\tau) h^*(-t+\tau) d\tau$, then

$$r[k] = r(kT) = \int y(\tau) h^*(\tau - kT) d\tau = \int y(t) h^*(t-kT) dt$$

Let $\rho(t) = h(t) * h^*(-t) \xrightarrow{\mathcal{F}} Q(f) = |H(f)|^2$, and $\rho[k] = \rho(kT)$, then

$$\rho(t-s) = \int h(\tau) h^*(\tau - t + s) d\tau = \int h(\mu - s) h^*(\mu - t) d\mu;\ \mu = \tau + s$$
$$\int h(t-kT) h^*(t-iT) dt = \rho(iT - kT) = \rho[i-k].$$

Thus,

$$\hat{S}_{ML} = \arg\min_{\vec{s}} -2\,\mathrm{Re}\left\{ \sum_{k=0}^{N-1} s^*[k] \int y(t) h^*(t-kT) dt \right\} + \left(\sum_{k=0}^{N-1} \sum_{i=0}^{N-1} s[k] s^*[i] \int h(t-kT) h^*(t-iT) dt \right)$$
$$= \arg\min_{\vec{s}} -2\,\mathrm{Re}\left\{ \sum_{k=0}^{N-1} s^*[k] r[k] \right\} + \left(\sum_{k=0}^{N-1} \sum_{i=0}^{N-1} s[k] s^*[i] \rho[i-k] \right)$$

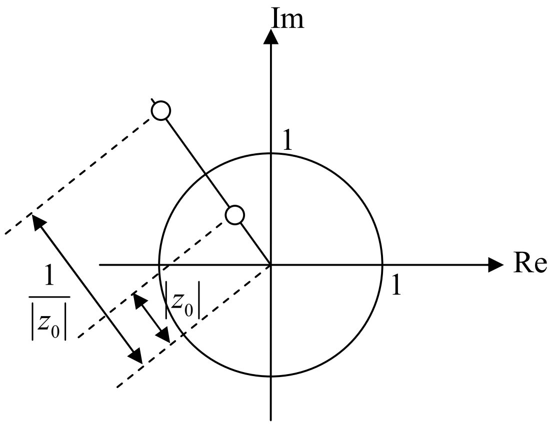
- Output $r[k]$ of the sampled matched filters are sufficient statistics.
- Detection can be made only after all $r[k]$ have been collected unless $\rho[k] = \rho[0]\delta[k]$.
  $\rho[0] = \int |h(t)|^2 dt$.
- The complexity grows exponentially with the number of states.
- Has long decision delay.
- Not easy for adaptive implementation.

- **Channel correlation function**
  - $\rho(t) = h(t) * h^*(-t) = \int h(\tau) h^*(\tau - t) d\tau$
  - $\rho(-t) = \rho^*(t)$

Proof $\rho(-t)=\int h(\tau)h^*(\tau+t)d\tau$

$$\rho^*(t)=\int h^*(\tau)h(\tau-t)d\tau=\int h^*(\mu+t)h(\mu)d\mu\,;\ \mu=\tau-t$$
$$=\int h(\tau)h^*(\tau+t)d\tau$$

- $\rho(0)=\int |h(\tau)|^2 d\tau$ (always real).
- $\rho(z)=\sum_{k=-\infty}^{\infty}\rho[k]z^{-k}$
- z_0 is a zero of $\rho(z)$ if and only if $\frac{1}{z_0^*}=\frac{1}{|z_0|}\left(\frac{z_0}{|z_0|}\right)$ is a zero of $\rho(z)$. z_0 and $\frac{1}{z_0^*}$ form a pair symmetrical with respect to the unit circle.



Proof $\rho(z)=\sum_{k=-\infty}^{\infty}\rho[k]z^{-k}=\rho[0]+\sum_{k=1}^{\infty}\rho[k]z^{-k}+\sum_{k=1}^{\infty}\rho[-k]z^{k}$

$$=\rho[0]+\sum_{k=1}^{\infty}\rho[k]z^{-k}+\sum_{k=1}^{\infty}\rho^*[k]z^{k}$$

z_0 is a zero of $\rho(z)\Rightarrow \rho[0]+\sum_{k=1}^{\infty}\rho[k]z_0^{-k}+\sum_{k=1}^{\infty}\rho^*[k]z_0^{k}=0$.

$$\rho\left(\frac{1}{z_0^*}\right)=\rho[0]+\sum_{k=1}^{\infty}\rho[k]z_0^{-k}+\sum_{k=1}^{\infty}\rho^*[k]z_0^{k}=\rho[0]+\sum_{k=1}^{\infty}\rho[k]\left(z_0^{k}\right)^*+\sum_{k=1}^{\infty}\rho^*[k]\left(z_0^{-k}\right)^*$$
$$=\left(\rho^*[0]+\sum_{k=1}^{\infty}\rho^*[k]\left(z_0^{k}\right)+\sum_{k=1}^{\infty}\rho[k]\left(z_0^{-k}\right)\right)^*$$
$$=0^*=0$$

$$\frac{1}{z_0^*}=\frac{1}{z_0^*}\frac{z_0}{z_0}=\frac{z_0}{|z_0|^2}=\frac{1}{|z_0|}\frac{z_0}{|z_0|}.$$

So, if z_0 is a zero of $\rho(z)$, so does $\frac{1}{z_0^*}$. Because $\frac{1}{\left(\frac{1}{z_0^*}\right)^*} = z_0$, the converse is also true, i.e., if $\frac{1}{z_0^*}$ is a zero of $\rho(z)$, so does z_0.

$\angle\left(\frac{1}{z_0^*}\right) = \angle z_0$ and $\left|\frac{1}{z_0^*}\right| = \frac{1}{|z_0|}$. So, $\frac{1}{z_0^*}$ is along the same line from origin as z_0, but its magnitude is $\frac{1}{|z_0|}$. Also, $\left|\frac{1}{z_0^*}\right| > 1 \Leftrightarrow |z_0| < 1$ and $\left|\frac{1}{z_0^*}\right| < 1 \Leftrightarrow |z_0| > 1$.

So, z_0 and $\frac{1}{z_0^*}$ form a pair symmetrical with respect to the unit circle.

- $\rho[k] = \rho(kT) = \int h(\tau) h^*(\tau - kT) d\tau$
- $\rho[-k] = \rho^*[k]$

  Proof $\rho[-k] = \rho(-kT) = \rho^*(kT) = \rho^*[k]$

The Nyquist Theorem

- **The Nyquist Theorem**: $p[k] = p[0]\delta[k]$ iff $\frac{1}{T}\sum_i Q\left(f - \frac{i}{T}\right) = \rho[0]$ and the MLSD is the same as symbol-by-symbol ML detection. $\hat{S}_{ML} = \arg\min_{\vec{s}}\left(\sum_{k=0}^{N-1}\left|s[k] - \frac{r[k]}{\rho[0]}\right|^2\right)$.

$\frac{1}{T}\sum_i Q\left(f - \frac{i}{T}\right) = \rho[0] \Rightarrow p[k] = p[0]\delta[k]$

Proof By the deconstruction equation, $\rho_{DTFT}(f) = \frac{1}{T}\sum_{n=-\infty}^{\infty} Q\left(\frac{f}{T} + \frac{n}{T}\right)$.

We are given that $\rho_{DTFT}(Tf) = \frac{1}{T}\sum_{n=-\infty}^{\infty} Q\left(f + \frac{n}{T}\right) = \rho[0]$. Thus, $\rho_{DTFT}(f) = \rho[0]$.

Using inverse DTFT, $\rho_{DTFT}(f) = \rho[0] \times 1 \xleftarrow{DTFT} \rho[k] = \rho[0]\delta[k]$.

$p[k] = p[0]\delta[k] \Rightarrow \frac{1}{T}\sum_i Q\left(f - \frac{i}{T}\right) = \rho[0]$

Proof $p[k] = p[0]\delta[k] \Rightarrow \rho_{DTFT}(f) = \frac{1}{T}\sum_{n=-\infty}^{\infty} Q\left(\frac{f}{T} + \frac{n}{T}\right) = \rho[0]$

$\Rightarrow \rho_{DTFT}(Tf) = \frac{1}{T}\sum_{n=-\infty}^{\infty} Q\left(f + \frac{n}{T}\right) = \rho[0]$.

$p[k] = p[0]\delta[k] \Rightarrow$ MLSD is the same as symbol by symbol ML detection.

Proof $\hat{S}_{ML} = \arg\min_{\vec{s}} -2\operatorname{Re}\left\{\sum_{k=0}^{N-1} s^*[k]r[k]\right\} + \left(\sum_{k=0}^{N-1}\sum_{i=0}^{N-1} s[k]s^*[i]\rho[i-k]\right)$

$$= \arg\min_{\vec{s}} -2\operatorname{Re}\left\{\sum_{k=0}^{N-1} s^*[k]r[k]\right\} + \left(\sum_{k=0}^{N-1} s[k]s^*[k]\rho[0]\right)$$

$$= \arg\min_{\vec{s}} -2\operatorname{Re}\left\{\sum_{k=0}^{N-1} s^*[k]r[k]\right\} + \left(\sum_{k=0}^{N-1} |s[k]|^2 \rho[0]\right)$$

$$= \arg\min_{\vec{s}} \left(\sum_{k=0}^{N-1}\left(|s[k]|^2 \rho[0] - 2\operatorname{Re}\{s^*[k]r[k]\}\right)\right)$$

Consider $|s[k]|^2 \rho[0] - 2\operatorname{Re}\{s^*[k]r[k]\}$,

$$|s[k]|^2 \rho[0] - 2\operatorname{Re}\{s^*[k]r[k]\}$$

$$= s^*[k]\sqrt{\rho[0]}s[k]\sqrt{\rho[0]} - s^*[k]\sqrt{\rho[0]}\frac{r[k]}{\sqrt{\rho[0]}} - s[k]\sqrt{\rho[0]}\frac{r^*[k]}{\sqrt{\rho[0]}}$$

$$= s^*[k]\sqrt{\rho[0]}\left(s[k]\sqrt{\rho[0]} - \frac{r[k]}{\sqrt{\rho[0]}}\right) - \frac{r^*[k]}{\sqrt{\rho[0]}}\left(s[k]\sqrt{\rho[0]} - \underbrace{\frac{r[k]}{\sqrt{\rho[0]}}}_{add}\right)$$

Because we are doing $\arg\min_{\vec{s}}$, the term $\frac{r[k]}{\sqrt{\rho[0]}}$ doesn't change the result.

Thus, $|s[k]|^2 \rho[0] - 2\operatorname{Re}\{s^*[k]r[k]\} = \left(s^*[k]\sqrt{\rho[0]} - \frac{r^*[k]}{\sqrt{\rho[0]}}\right)\left(s[k]\sqrt{\rho[0]} - \frac{r[k]}{\sqrt{\rho[0]}}\right)$

$$= \left|s[k]\sqrt{\rho[0]} - \frac{r[k]}{\sqrt{\rho[0]}}\right|^2$$

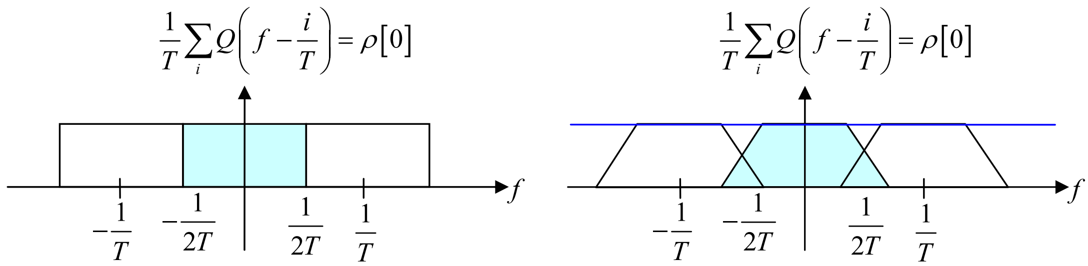
$$\hat{S}_{ML} = \arg\min_{\vec{s}}\left(\sum_{k=0}^{N-1}\left|s[k]\sqrt{\rho[0]} - \frac{r[k]}{\sqrt{\rho[0]}}\right|^2\right) = \arg\min_{\vec{s}}\left(\sum_{k=0}^{N-1}\left|s[k] - \frac{r[k]}{\rho[0]}\right|^2\right).$$

- The minimum bandwidth (max f – min f) required is $\frac{1}{T}$.

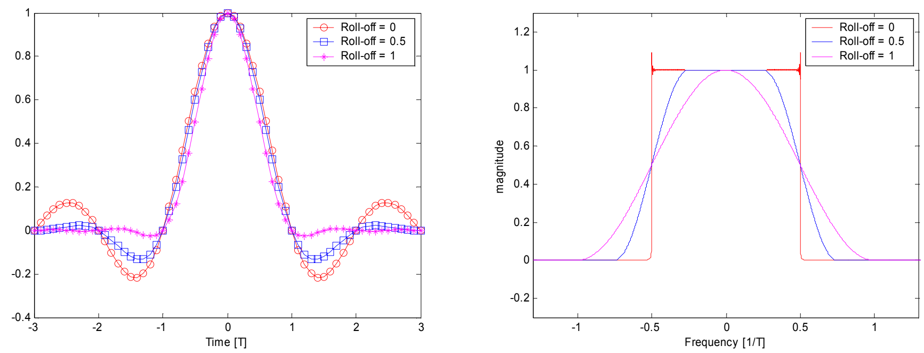
  Proof

  Need $\frac{1}{T}\sum_i Q\left(f - \frac{i}{T}\right) = \rho[0]$.

Because all the consecutive $Q\left(f-\frac{i}{T}\right)$ are $\frac{1}{T}$ apart. Each $Q\left(f-\frac{i}{T}\right)$ has to fill up the frequency $\frac{1}{T}$; otherwise, the will be gap between $Q\left(f-\frac{i}{T}\right)$'s which means the value of the $\frac{1}{T}\sum_i Q\left(f-\frac{i}{T}\right)$ is zero there. This can't be because we want $\frac{1}{T}\sum_i Q\left(f-\frac{i}{T}\right)=\rho[0]$.



- The matched filter implementation requires that we use square-root raised cosine function as the pulse-shaping filter.
- **Raised cosine pulse** with roll-off factor α
- $\rho_\alpha(t)=\frac{\sin\frac{\pi t}{T}}{\frac{\pi t}{T}}\frac{\cos\frac{\alpha\pi t}{T}}{1-\frac{4\alpha^2t^2}{T^2}}$.



- $$Q_\alpha(f)=\begin{cases} T, & 0\le|f|\le\frac{1-\alpha}{2T} \\ \frac{T}{2}\left(1+\cos\left(\frac{\pi T}{\alpha}\left(|f|-\frac{1-\alpha}{2T}\right)\right)\right), & \frac{1-\alpha}{2T}\le|f|\le\frac{1+\alpha}{2T} \\ 0, & |f|\ge\frac{1+\alpha}{2T} \end{cases}$$
- Bandlimited to $\frac{1+\alpha}{2T}$.

- $Q_\alpha(0) = T$, $Q_\alpha\left(\frac{1}{2T}\right) = \frac{T}{2}$
- $\rho[k] = \rho(kT) = \delta[k]$.

Viterbi Algorithm

- Given $\vec{r} = [r_0, r_1, \ldots, r_{N-1}]$, $\vec{\rho} = [\rho_0, \rho_1, \ldots, \rho_L]$, $\mathcal{C} = \{c_1, \ldots, c_c\}$.

  $\hat{s}_{ML} = \arg\min_{\vec{s}\in\mathcal{C}^N} C_{N-1}$ where

  Cost $C_t = -2\,\mathrm{Re}\left\{\sum_{\ell=0}^{t} s_\ell^* r_\ell\right\} + \left(\sum_{\ell=0}^{t}\sum_{m=0}^{t} s_\ell^* s_m \rho_{\ell-m}\right) = C_{t-1} + \Delta C(r_t, s_t; \pi_{t-1})$.

  $\Delta C(r_t, s_t; \pi_{t-1}) = -2\,\mathrm{Re}\left\{s_t^* r_t\right\} + 2\,\mathrm{Re}\left\{s_t^* \sum_{k=1}^{\min(t,L)} s_{t-k}\rho_k\right\} + |s_t|^2 \rho_0$

  Define the state at time *t* by $(s_t, \ldots, s_{t-L+1}) \Rightarrow \pi_t = \underbrace{(s_t, \ldots, s_{t-L+1})}_{L} \in \mathcal{C}^L$.

  For $t < L$, $\pi_t = (s_t, \ldots, s_0)$.

  $\pi_0 = s_0$. $C_0 = -2\,\mathrm{Re}\left\{s_0^* r_0\right\} + |s_0|^2 \rho_0$.

Idea:

$$\hat{S}_{ML} = \arg\min_{\vec{s}} - 2\,\mathrm{Re}\left\{\sum_{k=0}^{N-1} s^*[k] r[k]\right\} + \left(\sum_{k=0}^{N-1}\sum_{i=0}^{N-1} s[k] s^*[i] \rho[i-k]\right)$$

Define $C_t = -2\,\mathrm{Re}\left\{\sum_{\ell=0}^{t} s_\ell^* r_\ell\right\} + \left(\sum_{\ell=0}^{t}\sum_{m=0}^{t} s_\ell^* s_m \rho_{\ell-m}\right)$

Note that
$$\begin{aligned}\sum_{\ell=0}^{t}\sum_{m=0}^{t} s_\ell^* s_m \rho_{\ell-m} &= \left(\sum_{\ell=0}^{t-1}\sum_{m=0}^{t-1} s_\ell^* s_m \rho_{\ell-m}\right) + \left(s_t^* \sum_{m=0}^{t-1} s_m \rho_{t-m}\right) + \left(s_t \sum_{\ell=0}^{t-1} s_\ell^* \rho_{\ell-t}\right) + s_t^* s_t \rho_0 \\ &= \left(\sum_{\ell=0}^{t-1}\sum_{m=0}^{t-1} s_\ell^* s_m \rho_{\ell-m}\right) + \left(s_t^* \sum_{m=0}^{t-1} s_m \rho_{t-m}\right) + \left(s_t \sum_{m=0}^{t-1} s_m^* \rho_{t-m}^*\right) + s_t^* s_t \rho_0 \\ &= \left(\sum_{\ell=0}^{t-1}\sum_{m=0}^{t-1} s_\ell^* s_m \rho_{\ell-m}\right) + 2\,\mathrm{Re}\left\{s_t^* \sum_{m=0}^{t-1} s_m \rho_{t-m}\right\} + |s_t|^2 \rho_0 \\ &= \left(\sum_{\ell=0}^{t-1}\sum_{m=0}^{t-1} s_\ell^* s_m \rho_{\ell-m}\right) + 2\,\mathrm{Re}\left\{s_t^* \sum_{m=0}^{t-1} s_m \rho_{t-m}\right\} + |s_t|^2 \rho_0\end{aligned}$$

Consider $\sum_{m=0}^{t-1} s_m \rho_{t-m}$. Let $k = t - m$, then $2\,\mathrm{Re}\left\{s_t^* \sum_{m=0}^{t-1} s_m \rho_{t-m}\right\} = 2\,\mathrm{Re}\left\{s_t^* \sum_{k=1}^{t} s_{t-k}\rho_k\right\}$.

Now, $\rho_k \neq 0$ only for $-L \le k \le L$, thus $2\,\mathrm{Re}\left\{s_t^* \sum_{k=1}^{t} s_{t-k}\rho_k\right\} = 2\,\mathrm{Re}\left\{s_t^* \sum_{k=1}^{\min(t,L)} s_{t-k}\rho_k\right\}$.

We then have

$$\sum_{\ell=0}^{t}\sum_{m=0}^{t} s_\ell^* s_m \rho_{\ell-m} = \left(\sum_{\ell=0}^{t-1}\sum_{m=0}^{t-1} s_\ell^* s_m \rho_{\ell-m}\right) + 2\,\mathrm{Re}\left\{s_t^* \sum_{k=1}^{\min(t,L)} s_{t-k}\rho_k\right\} + \left|s_t\right|^2 \rho_0$$

And

$$\begin{aligned}
C_t &= -2\,\mathrm{Re}\left\{\sum_{\ell=0}^{t} s_\ell^* r_\ell\right\} + \left(\sum_{\ell=0}^{t}\sum_{m=0}^{t} s_\ell^* s_m \rho_{\ell-m}\right) \\
&= -2\,\mathrm{Re}\left\{\sum_{\ell=0}^{t-1} s_\ell^* r_\ell\right\} - 2\,\mathrm{Re}\left\{s_t^* r_t\right\} \\
&\quad + \left(\sum_{\ell=0}^{t-1}\sum_{m=0}^{t-1} s_\ell^* s_m \rho_{\ell-m}\right) + 2\,\mathrm{Re}\left\{s_t^* \sum_{k=1}^{\min(t,L)} s_{t-k}\rho_k\right\} + \left|s_t\right|^2 \rho_0 \\
&= \left\{-2\,\mathrm{Re}\left\{\sum_{\ell=0}^{t-1} s_\ell^* r_\ell\right\} + \left(\sum_{\ell=0}^{t-1}\sum_{m=0}^{t-1} s_\ell^* s_m \rho_{\ell-m}\right)\right\} \\
&\quad + \left\{-2\,\mathrm{Re}\left\{s_t^* r_t\right\} + 2\,\mathrm{Re}\left\{s_t^* \sum_{k=1}^{\min(t,L)} s_{t-k}\rho_k\right\} + \left|s_t\right|^2 \rho_0\right\} \\
&= C_{t-1} + \Delta C\left(r_t, s_t; \pi_{t-1}\right)
\end{aligned}$$

where $\Delta C\left(r_t, s_t; \pi_{t-1}\right) = -2\,\mathrm{Re}\left\{s_t^* r_t\right\} + 2\,\mathrm{Re}\left\{s_t^* \sum_{k=1}^{\min(t,L)} s_{t-k}\rho_k\right\} + \left|s_t\right|^2 \rho_0$.

So, we only need $\left\{s_{t-1},\ldots,s_{t-L}\right\}$ to define π_{t-1}.

Define the state at time *t* by $\left(s_t,\ldots,s_{t-L+1}\right) \Rightarrow \pi_t = \underbrace{\left(s_t,\ldots,s_{t-L+1}\right)}_{L} \in \mathcal{C}^L$.

For $t < L$, $\pi_t = \left(s_t,\ldots,s_0\right)$.

$\pi_0 = s_0$. $C_0 = -2\,\mathrm{Re}\left\{s_0^* r_0\right\} + \left|s_0\right|^2 \rho_0$.

- To calculate the cost:

Can start with the cost for state $\pi_{L-1} = \left(s_{L-1},\ldots,s_0\right)$. There are c^L possible states.

$$C_{L-1} = -2\,\mathrm{Re}\left\{\sum_{\ell=0}^{L-1} s_\ell^* r_\ell\right\} + \left(\sum_{\ell=0}^{L-1}\sum_{m=0}^{L-1} s_\ell^* s_m \rho_{\ell-m}\right).$$

Then, to find C_{N-1}, recursively use $C_t = C_{t-1} + \Delta C\left(r_t, s_t; \pi_{t-1}\right)$ where

$$\Delta C\left(r_t, s_t; \pi_{t-1}\right) = -2\,\mathrm{Re}\left\{s_t^* r_t\right\} + 2\,\mathrm{Re}\left\{s_t^* \sum_{k=1}^{L} s_{t-k}\rho_k\right\} + \left|s_t\right|^2 \rho_0.$$

Or, can start from $C_0 = -2\,\mathrm{Re}\left\{s_0^* r_0\right\} + \left|s_0\right|^2 \rho_0$.

For $L = 1$,

for $t > 0$, $\pi_t = s_t$: $C_t = C_{t-1} + \Delta C(r_t, s_t; \pi_{t-1})$

$$= C_{t-1} + \left(-2\,\mathrm{Re}\{s_t^* r_t\} + 2\,\mathrm{Re}\{s_t^* s_{t-1}\rho_1\} + |s_t|^2 \rho_0\right)$$

For $L = 2$,

$\pi_1 = (s_1, s_0)$: $C_1 = C_0 + \Delta C(r_1, s_1; \pi_0) = C_0 - 2\,\mathrm{Re}\{s_1^* r_1\} + 2\,\mathrm{Re}\{s_1^* s_0 \rho_1\} + |s_1|^2 \rho_0$.

for $t > 1$, $C_t = C_{t-1} + \Delta C(r_t, s_t; \pi_{t-1})$

$$\Delta C(r_t, s_t; \pi_{t-1}) = -2\,\mathrm{Re}\{s_t^* r_t\} + 2\,\mathrm{Re}\left\{s_t^* \sum_{k=1}^{2} s_{t-k}\rho_k\right\} + |s_t|^2 \rho_0$$

$$= -2\,\mathrm{Re}\{s_t^* r_t\} + 2\,\mathrm{Re}\{s_t^* s_{t-1}\rho_1 + s_t^* s_{t-2}\rho_2\} + |s_t|^2 \rho_0$$

- At each node, there should be only one surviving incoming path.
- Complexity is linear with respect to *N*.
- Ex: Real case: $\mathcal{C} = \{-1, 1\}$

k	0	1	2	3	4	5
$\rho[k]$	3	2	1	0	0	0
$r[k]$	-1	0	1	2	-2	1

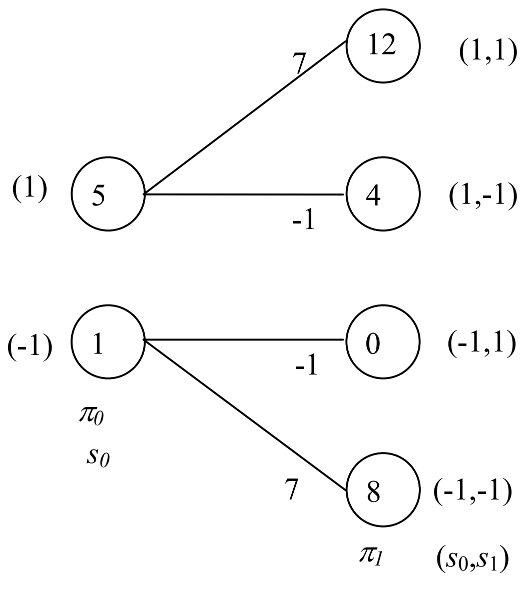
Here, $L = 2$, thus, $\pi_t = (s_t, s_{t-1}) \in \{(-1,-1), (-1,1), (1,-1), (1,1)\}$.

$C_0 = -2\,\mathrm{Re}\{s_0^* r_0\} + |s_0|^2 \rho_0 = 2s_0 + 3$.

$C_1 = C_0 + \Delta C(r_1, s_1; \pi_0)$

$= C_0 - 2\,\mathrm{Re}\{s_1^* r_1\} + 2\,\mathrm{Re}\{s_1^* s_0 \rho_1\} + |s_1|^2 \rho_0$

$= C_0 + 4 s_1 s_0 + 3$



For $t \geq L = 2$,

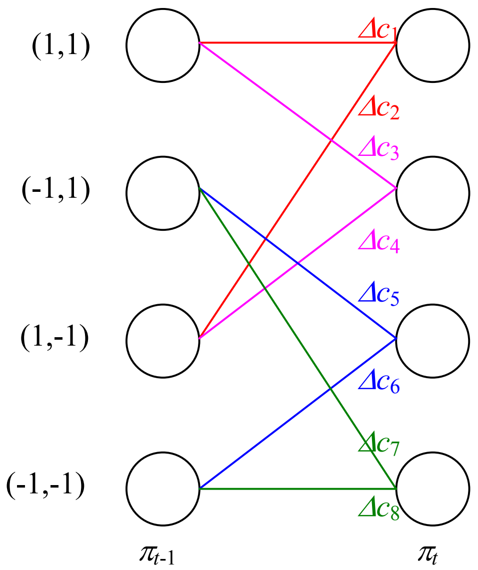
$$\Delta C(r_t, s_t; \pi_{t-1}) = -2\,\mathrm{Re}\{s_t^* r_t\} + 2\,\mathrm{Re}\left\{s_t^* \sum_{k=1}^{2} s_{t-k}\rho_k\right\} + |s_t|^2$$

$$= -2 s_t r_t + 2 s_t s_{t-1}\rho_1 + 2 s_t s_{t-2}\rho_2 + |s_t|^2 \rho_0$$

$$= -2 s_t r_t + 2 s_t s_{t-1}\rho_1 + 2 s_t s_{t-2}\rho_2 + \rho_0$$

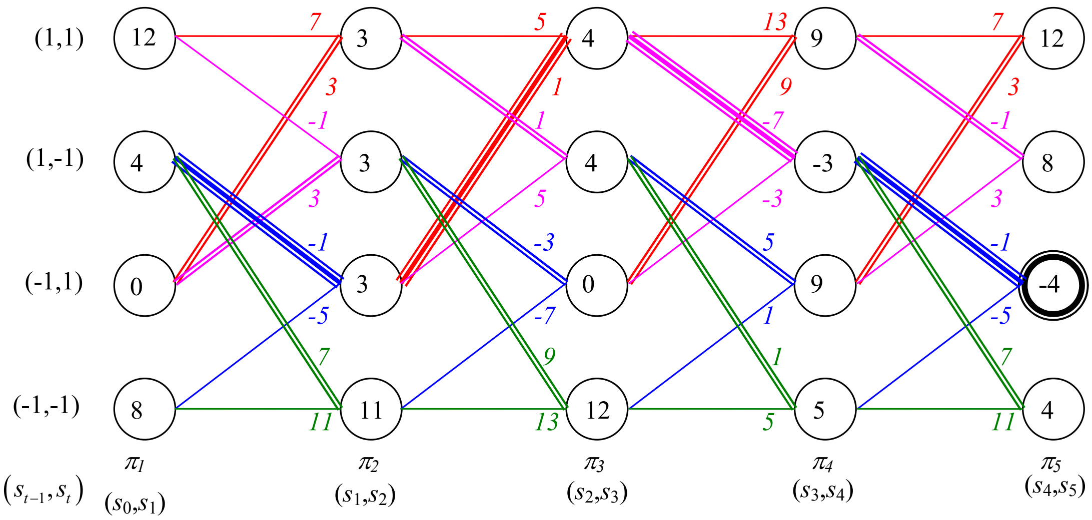
$$= -2 s_t r_t + 2 s_t s_{t-1} 2 + 2 s_t s_{t-2} 1 + 3$$

$$= -2 s_t r_t + 4 s_t s_{t-1} + 2 s_t s_{t-2} + 3$$



$$\Delta C_1 = \Delta C\left(s_t = 1, r_t; s_{t-1} = 1, s_{t-2} = 1\right) = -2r_t + 4 + 2 + 3 = 9 - 2r_t$$
$$\Delta C_2 = \Delta C\left(s_t = 1, r_t; s_{t-1} = 1, s_{t-2} = -1\right) = -2r_t + 4 - 2 + 3 = 5 - 2r_t$$
$$\Delta C_3 = \Delta C\left(s_t = -1, r_t; s_{t-1} = 1, s_{t-2} = 1\right) = 2r_t - 4 - 2 + 3 = -3 + 2r_t$$
$$\Delta C_4 = \Delta C\left(s_t = -1, r_t; s_{t-1} = 1, s_{t-2} = -1\right) = 2r_t - 4 + 2 + 3 = 1 + 2r_t$$
$$\Delta C_5 = \Delta C\left(s_t = 1, r_t; s_{t-1} = -1, s_{t-2} = 1\right) = -2r_t - 4 + 2 + 3 = 1 - 2r_t$$
$$\Delta C_6 = \Delta C\left(s_t = 1, r_t; s_{t-1} = -1, s_{t-2} = -1\right) = -2r_t - 4 - 2 + 3 = -3 - 2r_t$$
$$\Delta C_7 = \Delta C\left(s_t = -1, r_t; s_{t-1} = -1, s_{t-2} = 1\right) = 2r_t + 4 - 2 + 3 = 5 + 2r_t$$
$$\Delta C_8 = \Delta C\left(s_t = -1, r_t; s_{t-1} = -1, s_{t-2} = -1\right) = 2r_t + 4 + 2 + 3 = 9 + 2r_t$$

Trellis disgram:



k	0	1	2	3	4	5
$r[k]$	-1	0	1	2	-2	1
$\hat{s}[k]$	1	-1	1	1	-1	1

Error Probability of MLSD

- The probability of detecting the sequence correctly vanishes as the length increases.

- $K_1 Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right) \le P[\mathcal{E}] \le K_2 Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right) + o\left(Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right)\right)$ where $d_{\min}$ is the minimum distance between every pair of paths.
- One-shot transmission and detection: suppose that only a single symbol s is transmitted over a Baseband channel $h(t)$ modeled by $r(t) = sh(t) + n(t)$ where $n(t)$ is the complex Gaussian noise with PSD N_0. $K_1 Q\left(\sqrt{\frac{\xi_{\min}^2 E_h}{2N_0}}\right) \le P[\mathcal{E}] \le K_2 Q\left(\sqrt{\frac{\xi_{\min}^2 E_h}{2N_0}}\right)$ where $E_h = \int |h(t)|^2 dt$ is the channel energy, and $\xi_{\min}$ is the minimum distance of the symbol constellation.
- Define $\gamma_{MF} = \frac{\xi_{\min}^2 E_h}{2N_0}$, $\gamma_{MLSD} = \frac{d_{\min}^2}{2N_0}$ where $d_{\min}$ is the minimum distance of any pair of sequences.
  - $\gamma_{MLSD} \le \gamma_{MF}$
  - $Q\left(\sqrt{\gamma_{MLSD}}\right) \ge Q\left(\sqrt{\gamma_{MF}}\right)$ because Q is a decreasing function.

Fourier Transform, DTFT (1)

- $\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{X}(\Omega) e^{j\Omega t} d\Omega = x(t) \underset{\mathfrak{I}^{-1}}{\overset{\mathfrak{I}}{\rightleftharpoons}} \hat{X}(\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt$.
- $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{X}(\omega) e^{jn\omega} d\omega \overset{DTFT}{\rightleftharpoons} \hat{X}(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-jn\omega}$.
  - $\hat{X}(\omega + 2k\pi) = \hat{X}(\omega)$.
- Sampling
  - $x[n] = x_c(nT_S);\ -\infty < n < \infty$
  - Deconstruction equation: $\hat{X}(\omega) = \sum_{k=-\infty}^{\infty}\left(\frac{1}{T_S}\hat{X}_c\left(\frac{\omega}{T_S} + k\frac{2\pi}{T_S}\right)\right)$ $\forall\, \omega, T_S$

    (Oppenheim, Eqn 4.53)
  - $x[n] = x_c(nT) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{\left\{\sum_{k=-\infty}^{\infty}\left(\frac{1}{T}\hat{X}_c\left(\frac{\omega}{T} + k\frac{2\pi}{T}\right)\right)\right\}}_{\hat{X}(\omega)} e^{jn\omega} d\omega$

Fourier Transform, DTFT (2)

- $\int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df = x(t) \underset{\mathfrak{I}^{-1}}{\overset{\mathfrak{I}}{\rightleftharpoons}} X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$
  - $X(f) = \hat{X}(\Omega)\Big|_{\Omega = 2\pi f}$, $\hat{X}(\Omega) = X(f)\Big|_{f = \frac{\Omega}{2\pi}}$
  - $x^*(t) \overset{\mathcal{F}}{\longleftrightarrow} X^*(-f)$

Proof Let $y(t) = x^*(t)$. Then,

$$Y(f) = \int_{-\infty}^{\infty} y(t) e^{-j2\pi ft} dt = \int_{-\infty}^{\infty} x^*(t) e^{-j2\pi ft} dt = \int_{-\infty}^{\infty} x^*(t) e^{j2\pi(-f)t} dt$$
$$= \left(\int_{-\infty}^{\infty} x(\tau) e^{-j2\pi(-f)\tau} d\tau \right)^*$$
$$= X^*(-f)$$

- $x^*(-t) \xleftrightarrow{\mathcal{F}} X^*(f)$

Proof Let $y(t) = x^*(-t)$. Then,

$$Y(f) = \int_{-\infty}^{\infty} y(t) e^{-j2\pi ft} dt = \int_{-\infty}^{\infty} x^*(-t) e^{-j2\pi ft} dt = \int_{-\infty}^{\infty} x^*(\tau) e^{j2\pi(-f)(-\tau)} d\tau$$
$$= \left(\int_{-\infty}^{\infty} x(\tau) e^{-j2\pi f\tau} d\tau \right)^*$$
$$= X^*(f)$$

- $x[n] = \int_{-\frac{1}{2}}^{\frac{1}{2}} \hat{X}_{DTFT}(f) e^{jn2\pi f} df \xrightleftharpoons{DTFT} \hat{X}_{DTFT}(f) = \sum_{n=-\infty}^{\infty} x[n] e^{-jn2\pi f}$
  - $\hat{X}_{DTFT}(f) = \hat{X}(\omega)\Big|_{\omega=2\pi f}$
- Sampling
  - $x[k] = \int_{-\frac{1}{2}}^{\frac{1}{2}} \underbrace{\frac{1}{T} \sum_{n=-\infty}^{\infty} X\left(\frac{f}{T} + \frac{n}{T}\right)}_{DTFT} e^{j2\pi fk} df$.
  - $X_{DTFT}(f) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X\left(\frac{f}{T} + \frac{n}{T}\right)$.

Proof1 $\hat{X}(\omega) = \sum_{k=-\infty}^{\infty} \left(\frac{1}{T_S} \hat{X}_c\left(\frac{\omega}{T_S} + k\frac{2\pi}{T_S} \right) \right) = \sum_{k=-\infty}^{\infty} \left(\frac{1}{T_S} X\left(\frac{\omega}{2\pi T_S} + \frac{k}{T_S} \right) \right)$ because $\hat{X}(\Omega) = X(f)\Big|_{f=\frac{\Omega}{2\pi}}$

$$X_{DTFT}(f) = \hat{X}(\omega)\Big|_{\omega=2\pi f} = \sum_{k=-\infty}^{\infty} \left(\frac{1}{T_S} X\left(\frac{2\pi f}{2\pi T_S} + \frac{k}{T_S} \right) \right) = \sum_{k=-\infty}^{\infty} \left(\frac{1}{T_S} X\left(\frac{f}{T_S} + \frac{k}{T_S} \right) \right).$$

Proof2 $x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df = \sum_{n=-\infty}^{\infty} \int_{\frac{n}{T}-\frac{1}{2T}}^{\frac{n}{T}+\frac{1}{2T}} X(f) e^{j2\pi ft} df$.

Let $\mu = f - \frac{n}{T}$. Then,

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df = \sum_{n=-\infty}^{\infty} \int_{-\frac{1}{2T}}^{\frac{1}{2T}} X\left(\mu + \frac{n}{T}\right) e^{j2\pi\left(\mu + \frac{n}{T}\right)t} d\mu = \sum_{n=-\infty}^{\infty} \int_{-\frac{1}{2T}}^{\frac{1}{2T}} X\left(\mu + \frac{n}{T}\right) e^{j2\pi\mu t} e^{j2\pi\frac{n}{T}t} d\mu$$

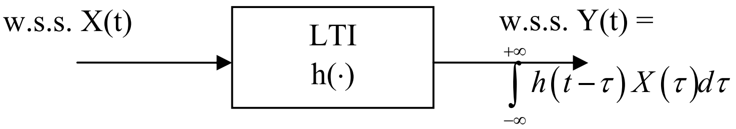
$$x[k] = x(kT) = \sum_{n=-\infty}^{\infty} \int_{-\frac{1}{2T}}^{\frac{1}{2T}} X\left(\mu + \frac{n}{T}\right) e^{j2\pi\mu kT} \cancel{e^{j2\pi\frac{n}{T}kT}}^{1} d\mu = \int_{-\frac{1}{2T}}^{\frac{1}{2T}} \sum_{n=-\infty}^{\infty} X\left(f + \frac{n}{T}\right) e^{j2\pi fkT} df$$

Let $\mu = fT$. Then,

$$x[k] = \frac{1}{T} \int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{n=-\infty}^{\infty} X\left(\frac{\mu}{T} + \frac{n}{T}\right) e^{j2\pi\mu k} d\mu = \int_{-\frac{1}{2}}^{\frac{1}{2}} \underbrace{\frac{1}{T} \sum_{n=-\infty}^{\infty} X\left(\frac{f}{T} + \frac{n}{T}\right)}_{DTFT} e^{j2\pi fk} df .$$

$$X_{DTFT}(f) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X\left(\frac{f}{T} + \frac{n}{T}\right).$$

Power Spectral Density



- $R_Y(\tau) = R_X(\tau) * h(\tau) * h^*(-\tau)$

Proof $\quad y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(t-\mu) h(\mu) d\mu$

$$y(t-\tau) = \int_{-\infty}^{\infty} x(t-\tau-\mu') h(\mu') d\mu' = \int_{-\infty}^{\infty} x(t-(\tau+\mu')) h(\mu') d\mu'$$

$$y^*(t-\tau) = \int_{-\infty}^{\infty} x^*(t-(\tau+\mu')) h^*(\mu') d\mu'$$

$$R_Y(\tau) = E\left[y(t) y^*(t-\tau)\right] = E\left[\left(\int_{-\infty}^{\infty} x(t-\mu) h(\mu) d\mu\right)\left(\int_{-\infty}^{\infty} x^*(t-(\tau+\mu')) h^*(\mu') d\mu'\right)\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E\left[x(t-\mu) x^*(t-(\tau+\mu'))\right] h(\mu) h^*(\mu') d\mu' d\mu$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_X(\tau+\mu'-\mu) h(\mu) h^*(\mu') d\mu' d\mu$$

$$R_X(\tau)*h(\tau)*h^*(-\tau)=\left(\int_{-\infty}^{\infty}R_X(\tau-\mu)h(\mu)\right)*h^*(-\tau)$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}R_X(\tau-\mu'-\mu)h(\mu)h^*(-\mu')d\mu'd\mu$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}R_X(\tau+\mu''-\mu)h(\mu)h^*(\mu'')d\mu''d\mu$$

Thus, $R_Y(\tau)=R_X(\tau)*h(\tau)*h^*(-\tau)$.

- $S_Y(f)=S_X(f)\left|H(f)\right|^2$

  Proof $R_Y(\tau)=R_X(\tau)*h(\tau)*h^*(-\tau)$.

  Thus, $S_Y(f)=S_X(f)H(f)H^*(f)=S_X(f)\left|H(f)\right|^2$.

- $h(\tau)*h^*(-\tau)=\int_{-\infty}^{+\infty}h(\mu)h^*(\mu-\tau)d\mu$

  Proof $h(\tau)*h^*(-\tau)=\int_{-\infty}^{+\infty}h(\mu)h^*(-(\mu-\tau))d\mu=\int_{-\infty}^{+\infty}h(\mu)h^*(-(\tau-\mu))d\mu=\int_{-\infty}^{+\infty}h(\mu)h^*(\mu-\tau)d\mu$

- Let $y[k]=y(t)\big|_{t=kT}$ and $R_y[k]=E\left[y[m]y^*[m-k]\right]$, then $R_y[k]=R_y(kT)$

  Proof $R_y[k]=E\left[y[m]y^*[m-k]\right]=E\left[y(mT)y^*(mT-kT)\right]=R_y(kT)$

- If $R_X(\tau)=N_0\delta(\tau)$, then
  - $R_Y(\tau)=N_0\left(h(\tau)*h^*(-\tau)\right)=N_0\int_{-\infty}^{+\infty}h(\mu)h^*(\mu-\tau)d\mu$.
  - $R_y[k]=R_y(kT)=N_0\int_{-\infty}^{+\infty}h(\mu)h^*(\mu-kT)d\mu=N_0\left(h(t)*h^*(-t)\right)\Big|_{t=kT}$

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